arXiv:math/0608226v3 [math.CV] 15 Apr 2008

BERGMAN KERNELS AND EQUILIBRIUM MEASURES FOR POLARIZED PSEUDOCONCAVE DOMAINS

ROBERT BERMAN

ABSTRACT. Let X be a domain in a closed polarized complex manifold (Y, L), where L is a (semi-)positive line bundle over Y. Any given Hermitian metric on L induces by restriction to X a Hilbert space structure on the space of global holomorphic sections on Y with values in the kth tensor power of L (also using a volume form ω_n on X). In this paper the leading large k asymptotics for the corresponding Bergman kernels and metrics are obtained in the case when X is a pseudoconcave domain with smooth boundary (under a certain compatibility assumption). The asymptotics are expressed in terms of the curvature of L and of the boundary of X. The convergence of the Bergman metrics is obtained in a very general setting where (X, ω_n) is replaced by any measure satisfying a Berstein-Markov property. As an application the (generalized) equilibrium measure of the polarized pseudoconcave domain X is computed explicitely. Other applications to the zero and mass distribution of random holomorphic sections and the eigenvalue distribution of Toeplitz operators will appear elsewhere.

CONTENTS

1. INTRODUCTION

Let L be a holomorphic line bundle over a closed (i.e. compact without boundary) projective complex manifold Y of dimension n. Denote by $H^0(Y, L^k)$ the vector space of all global holomorphic sections on Y with

Key words and phrases. Line bundles, holomorphic sections, Bergman kernel asymptotics, global pluripotential theory, orthogonal polynomials *MSC (2000):* 32A25, 32L10, 32L20, 32U15, 42C05.

values in the kth tensor power of L. Any given Hermitian metric ϕ on L and a domain X in Y together with a volume form ω_n induces an L^2-norm on $H^0(Y, L^k)$ obtained by integrating the point-wise norms of sections in $H^0(X, L^k)$ over the domain X. The corresponding Hilbert space will be denoted by $\mathcal{H}(Y, L^k)_X$. The *Bergman kernel* $K^k(x, y)$ of the Hilbert space $\mathcal{H}(Y, L^k)_X$ is the integral kernel of the orthogonal projection from the space of all smooth sections with values in L^k onto $\mathcal{H}(Y, L^k)_X$.

In this paper the situation when the curvature form $dd^c\phi$ is *semi-positive* and the domain $X = \{\rho \leq 0\}$ has a smooth strictly pseudo-concave boundary, i.e. the Levi curvature form $dd^c\rho$ of the boundary is *negative,* will be mainly investigated. Then X (or rather the triple (X, L, ϕ)) will be called a *polarized pseudo-concave domain.*

In the case when $X = Y$ and the curvature form $dd^c\phi$ is *positive* the asymptotics of the Bergman kernel $K_k(x, y)$ as k tends to infinity have been studied extensively [35, 38, 1, 8] and are by now very well-understood due to strong locazation properties. For example, in scaled coordinates on "length-scales" of the order $1/k^{1/2}$ the Bergman kernels $K_k(x, y)$ converge (with all derivatives) to constant curvature model kernels. In particular, the leading asymptotics of the *Bergman measure* $B^k\omega_n$, where $B^k(y) := \left|K^k(y, y)\right|^2_{k\phi}$ (the point-wise norm) may be expressed in terms of the local curvature of L :

$$k^{-n}B^k\omega_n \to (dd^c\phi)^n/n! \tag{1.1}$$

uniformly on Y. As an immediate consequence Tian's almost isometry theorem [35] holds

$$k^{-1}\Omega_k := k^{-1}dd^c\ln K^k(y, y) \to dd^c\phi \tag{1.2}$$

uniformly on Y, where $k^{-1}\Omega_k$ is called the (normalized) k th *Bergman metric on* Y. Note that the latter asymptotics are considerably weaker than 1.1.

One notable application of these asymptotics was introduced by Shiffman-Zelditch in their study of random zeroes of random and quantum chaotic holomorphic sections [31] (see section 1.2 below) and was further developed in a series of papers (for example with Bleher [11, 12]).

A concrete realization of the situation studied in this paper is obtained by taking Y as the $n-$dimensional projective space $\mathbb{P}^n$ and L as the hyperplane line bundle $\mathcal{O}(1)$. Then the Hilbert space $\mathcal{H}(Y, L^k)_X$ may be identified with the space of all polynomials $p_k(z)$ in $\mathbb{C}^n$ of total degree at most k, equipped with the weighted norm

$$\|p_k\|^2_{k\phi,X} := \int_X |p_k(z)|^2 e^{-k\phi(z)}\omega_n, \tag{1.3}$$

where X has been replaced by its restrictio to the affine piece $\mathbb{C}^n$, where ϕ is a smooth plurisubharmonic function of logarithmic growth and ω_n is the restricted Fubini-Study volume form (then the integrals are finite). Moreover, X is by assumption the complement of a bounded pseudoconvex domain in $\mathbb{C}^n$. Such "weighted polynomials" (with ω_n replaced by a

measure supported on a "arbitrary" set X) have been recently studied in various contexts. See for example [16] and Bloom's appendix in the book [30] by Saff-Totik and the book [19] by Deift for the case when E is a set in $\mathbb{E}$, concerning relations to (hermitian) random matrix theory. Very recently Bloom-Shiffman [18] studied the "unweighted" situation obtained by setting $\phi = 0$ in 1.3 and letting X be a "regular" bounded set in $\mathbb{C}^n$ (see section 7.2). Using pluripotential theory [26] it was shown in [18] that the corresponding normalized k th "Bergman volume form" (compare formula 1.2) converges weakly to the *equilibrium measure* μ_e of X, supported on the (Shilov-) boundary of X :

$$(\Omega_k/k)^n/n! \to \mu_e \tag{1.4}$$

When the domain X is polarized (i.e. $dd^c\phi > 0$) the situation in the *interiour* of X can be shown to localize (as in 1.1). The main purpose of the present paper is to study the influence of the *boundary* on the Bergman kernel asymptotics of $\mathcal{H}(Y, L^k)_X$ and on a generalized equilibrium measure of the polarized pseudoconcave domain X (defined following the very recent work [25] of Guedj-Zeriahi). In the situation of Shiffman-Bloom refered to above these objets may, in general, not be expressed in terms of the local curvature of the boundary ∂X of the domain X. However, under the assumption of global negativitity of the curvature of the boundary ∂X there is a natural locally defined candidate for the boundary contribution, namely the following $2n-1$ form, invariantly defined on the boundary of X :

$$\mu := \int_0^T (dd^c\phi + tdd^c\rho)^{n-1} \wedge d^c\rho)dt/(n-1)!, \tag{1.5}$$

where T is the following function on ∂X, that will be referred to as the *slope function*:

$$T = \sup\left\{t \geq 0 : \, (dd^c\phi + tdd^c\rho)_x \geq 0 \, \text{along} \, T^{1,0}(\partial X)_x\right\}.$$

The point is that T is finite when ∂X is pseudoconcave. It will be shown that, further assuming a certain *compatibility* between the curvature $dd^c\rho$ of the boundary ∂X and the curvature $dd^c\phi$ of line bundle L, leads to localization properties of the Bergman kernel asymptotics and the (generalized) equilibrium measure. In fact, as illustrated by the examples in section 3.3, there are large classes of polarized pseudo-concave domains X where the localization properties hold precisely when the assumption on "compatible curvatures" holds.

The main results below are based on the Bergman kernel asymptotics obtained in section 5. A major role in the proofs of these asymptotics is played by the local holomorphic Morse inequalities obtained in [2, 4]. In the present setting these inequalities can be seen as refined versions of the Bernstein-Markov inequalities used by Shiffman-Bloom (compare section 7.3). In the last section some open problems concerning general smooth domains X (and even more general situations) are formulated.

These open problems should be seen in the light of some very recent developments that have appeared since the preprint of the first version of the present paper appeared: in [6] the situation when $X = Y$, but the curvature of L is arbitrary is studied and in [16, 17] the planar case is studied.

Finally we turn to the precise statement of the main results (see section 2 for further notation).

1.1. **Overview of the main present results.** The polarized pseudoconcave domain X will be said to have "compatible curvatures" when the slope function T above is constant on ∂X for some choice of the defining function ρ and certain further assumptions depending on the "filling" $Y - X$ of X hold (see section 2.2). For example, in the case of polynomials refered to above the compatibility assumption holds if $-\rho = \phi$ in 1.3.

Bergman kernel asymptotics (section 5). The first main result gives the convergence as a measure of the Bergman kernel:

Theorem 1.1. *Let K^k be the Bergman kernel for the Hilbert space $\mathcal{H}(Y, L^k)_X$ associated to the polarized pseudoconcave domain X with compatible curvatures. Denote by $\Delta_{X \times X}$ and $\Delta_{\partial X \times \partial X}$ the currents of integration on the diagonal in $X \times X$ and $\partial X \times \partial X$, respectively. Then the sequence of measures*

$$k^{-n} \left| K^k(x, y) \right|^2_{k\phi} 1_X(x) \omega_n(x) \wedge 1_X(y) \omega_n(y)$$

converges on $Y \times Y$ to

$$[\Delta_{X \times X}] \wedge 1_{X(0)} (dd^c \phi)^n / n! + [\Delta_{\partial X \times \partial X}] \wedge \mu$$

*in the weak *-topology, where μ is the $2n - 1$ form 1.5 on ∂X.*

In fact, in order to prove the previous theorem the following "special case" will first be shown for the corresponding Bergman measure (compare formula 1.6):

$$k^{-n} B^k 1_X \omega_n \to 1_X (dd^c \phi)_n + [\partial X] \wedge \mu \tag{1.6}$$

weakly as measures on Y. The next theorem concerns the scaling convergence of the Bergman kernel K^k close to the diagonal. It shows that after scaling K^k converges to constant curvature model kernels (at least after choosing a subsequence). The scalings are expressed in terms of the "normal" local coordinates introduced in section 4.1 and 4.3, respectively. In the statement below the dependence on the fixed center (which is the point x in the interior and the point σ at the boundary) has been suppressed.

Theorem 1.2. *Let K^k be the Bergman kernel for the Hilbert space $\mathcal{H}(Y, L^k)_X$ associated to the polarized pseudoconcave domain X with compatible curvatures. K^k has a subsequence K^{k_j} such that for almost any point x in the interior of X (i.e. $x \in X - E$, where E has measure*

zero) the following scaling asymptotics hold in the $\mathcal{C}^\infty-$topology on any compact subset of $\mathbb{C}^n_z \times \mathbb{C}^n_{z'}$:

$$(i)\, k_j^{-n} K^{k_j}(z/k_j^{1/2}; z'/k_j^{1/2}) \to K^0(z;z'),$$

where K^0 is the corresponding model Bergman kernel (formula 4.6). Moreover, for almost any fixed point σ in the boundary ∂X (i.e. $\sigma \in \partial X - F$, where F has measure zero in ∂X) the following scaling asymptotics hold in the $\mathcal{C}^\infty-$topology on any compact subset of $\mathbb{C}^n_{z,w} \times \mathbb{C}^n_{z',w'}$:

$$(ii)\, k_j^{-(n+1)} K^{k_j}(z/k_j^{1/2}, w/k_j; z'/k_j^{1/2}, w'/k_j) \to K^0(z,w;z',w'),$$

where K^0 is the corresponding model Bergman kernel (formula 4.15). Furthermore, the same statement holds after replacing K^k with any subsequence K^{k_l} (a priori E and F then depend on the subsequence K^{k_l}).

The model kernel K^0 associated to a point in the boundary may be expreesed by the following suggestive formula, where ρ_0 denotes the (polarized) defining function of the corresponding constant curvature model domain:

$$K^0 = \frac{1}{4\pi}\frac{1}{\pi}\det(dd^c\rho_0)e^{\phi_0}P(\frac{\partial}{\partial\rho_0})\frac{\partial}{\partial\rho_0}(\frac{e^{T\rho_0}-1}{\rho_0}),$$

where P is the characteristic polynomial of the linear operator $\{dd^c\phi\}_x\{-dd^c\rho\}_x^{-1}$. This kernel should be compared with the one obtained by Shiffman-Zelditch [32] in the *one*-dimensional unweighted case refered to above (the later kernel is essentially given by $\frac{e^v-1}{v}$ in special coordinates). The proofs in [32] relied on classical results of Carleman concerning the corresponding orthogonal polynomials and the exterior Riemann mapping theorem. The corresponding unweighted higher-dimensional scaling result in $\mathbb{C}^n$ was stated as an open problem in [18].

Bergman metric asymptotics (section 6). Denote by F_k the interior scaling maps on $\mathbb{C}^n$, as well as the boundary ones, corresponding to the scaling of the coordinates in theorem 1.2 above. The following theorem gives the convergence of the kth Bergman metric on Y induced by the polarized pseudoconcave domain X (compare section 6 for definitions).

Theorem 1.3. *Let Ω_k be the Bergman metric on Y induced by the polarized domain X with compatible curvatures. Then the following convergence holds for the corresponding normalized volume form:*

$$(\Omega_k/k)_n \to 1_X(dd^c\phi)_n/n! + [\partial X]\wedge\mu,$$

when k tends to infinity, as measures in the weak-topology, where μ is the $2n-1$ form 1.5 on ∂X.*

Moreover, the following scaling asymptotics for the pth exterior power of Ω_k hold (after replacing K_k with a subsequence as in theorem 1.2) around almost any interior point:

$$(i)\, F_k^*(\Omega_k)^p \to (dd^c\phi)^p$$

(with uniform convergence on each compact set) and around almost any boundary point:

$$(ii)\, F_k^*(\Omega_k)^p \to (dd^c\phi + tdd^c\rho + dt \wedge d^c\rho)^p$$

(with uniform convergence on each compact set), where t is the following function of $\rho : t = \frac{\partial}{\partial \rho} ln B^0(\rho)$ (see formula 4.13) so that $t(-\infty) = 0$ and $t(\infty) = T$ (where T is the slope function in formula 1.5).

Equilibrium measures (section 7). Following the recent work [25] of Guedj-Zeriahi (see also [16] for the weighted case in $\mathbb{C}^n$) let now X be any compact set in X and ϕ the "restriction" to X of a continuous metric on L. The corresponding *equilibrium metric* on $L \to Y$ is defined by

$$\phi_e(y) = \sup \left\{ \widetilde{\phi}(y) : \widetilde{\phi} \in \mathcal{L}_{(X,L)},\, \widetilde{\phi} \leq \phi \text{ on } X \right\}. \tag{1.7}$$

where $\mathcal{L}_{(X,L)}$ is the class consisting of all (possibly singular) metrics on L with positive curvature current. Consider the "regular" case when ϕ_e is in $\mathcal{L}_{(X,L)}$ (compare section 7.2). The Monge-Ampere measure $(dd^c\phi_e)^n/n!$ is called the *equilibrium measure* associated to (X, ϕ). It was recently introduced in the more general global setting of quasiplurisubharmonic functions by Guedj-Zeriahi [25], building on the work of Bedford-Taylor, Demailly and others. The item (i) in the following theorem implies that if Y is *any* smooth domain then the normalized k th Bergman volume form converges to the corresponding equilibrium measure (see section 7.3 for the definition of Bernstein-Markov measures etc). In item (ii) the optimal rate of convergence (saturated by the model examples in section 3 - see [4]) is obtained in the case when Y is strongly pseudoconcave.

Theorem 1.4. *Let X be a compact set in Y and ϕ a continous metric on an ample line bundle $L \to Y$.*

(i) Let ω_n be a volume form on Y. If $1_X\omega_n$ has the Bernstein-Markov property w.r.t (X, ϕ), then the following uniform convergence holds on all of Y :

$$k^{-1} ln\, K^k(y, y) \to \phi_e(y) \tag{1.8}$$

where K^k is the Bergman kernel associated to (X, ω_n, ϕ). In particular, the equilibrium metric ϕ_e is continuous then, i.e. (X, ϕ) is regular then.

(ii) If furthermore X is assumed to be a pseudoconcave domain with smooth boundary and ϕ is smooth, then the rate of the convergence in 1.8 is of the order $(n + 1)\ln k/k$.

(iii) If ν is any fixed measure which has the Bernstein-Markov property w.r.t (X, ϕ) and (X, ϕ) is regular, then the uniform convergence 1.8 holds for the Bergman kernel associated to (X, ν, ϕ).

Moreover, if L is only assumed to be a semi-positive line bundle, then the following convergence holds under any of the assumptions (i), (ii) or (iii) above for the normalized volume form of the corresponding kth

Bergman metric Ω_k :

$$(\Omega_k/k)^n \to (dd^c\phi_e)^n \tag{1.9}$$

when k *tends to infinity, as measures in the weak*-topology.*

The theorem above generalizes the result 1.4 of Shiffman-Bloom, concerning the unweighted case in $\mathbb{C}^n$ (as well as Theorem 2.1 in [14] concerning the weighted case for X a compact set in $\mathbb{C}^n$). The proof is similar to Demailly's $\overline{\partial}-$proof of Siciak's fundamental convergence result for the $L^\infty-$ version of the Bergman metrics (compare remark 7.4) in the unweighted case in $\mathbb{C}^n$ [22]. See also [25] for the global *polarized* case of this $L^\infty-$ version of the result. Also note that in the case when $X=Y$ the convergence towards the equilibrium measure was obtained in [6] for *any* line bundle L (when suitably formulated). The proof of the lower bound in the convergence of the theorem above uses the Ohsawa-Takegoshi extension theorem, which allows a precise controle on the rate of the convergence.

In case X is a polarized pseudoconcave domain that satisfies the assumption about compatible curvatures (section 2.2) the equilibrium measure can now be computed explicitly using theorem 1.3 (without assuming that L is ample):

Corollary 1.5. *Let* X *be a polarized pseudoconcave domain with compatible curvatures (section 2.1).* Then the (generalized) equilibrium measure $(dd^c\phi_e)^n/n!$ of the polarized domain X is given by

$$(dd^c\phi_e)/n! = 1_X(dd^c\phi)^n/n! + [\partial X]\wedge\mu,$$

where μ *is the* $2n-1$ *form 1.5 on* ∂X*.*

1.2. **Relations to random sections and Toeplitz operators.** In a sequel [7] to his paper the present results will be applied to the study of various random measure processes. The starting point is that any Hilbert space $\mathcal{H}_k$ (here $\mathcal{H}(Y,L^k)_X$) comes equipped with a natural Gaussian probability measure. As shown by Shiffman-Zelditch the Berfman measure $k^{-n}B^k\omega_n$ (formula 1.6) then represents the *expected mass distribution* $\mathbb{E}(|f_k|^2\,\omega_n)$ of a random section f_k in $\mathcal{H}_k$ and the Bergman volume form $(dd^c(\ln K^k(z,z)))^n/n!$ represents the *expexted distribution of simultaneous zeroes of* n *random sections* in $\mathcal{H}_k$. Moreover, the variance of the mass distribution can to the leader order be expressed in terms of the eigenvalue distribution of Toeplitz operators acting on $\mathcal{H}_k$ (compare [31]), which in turn may be obtained from the weak convergence of the measure $k^{-n}\left|K^k(x,y)\right|^2_{k\phi}\omega_n(x)\wedge\omega_n(y)$. Furthermore, the scaling properties of the Bergman kernel $K^k(x,y)$ are used to express the limit correlations between random zeroes (compare [11, 12]). In [7] the non-local effects appearing when the condition about "compatible curvatures" does not hold will also be investigated and related to the situation studied in [6], as well as the Hele-Shaw flow in interface dynamics (also called Laplacian growth) [37].

2. Setup

2.1. **Notation.** Let L be an Hermitian holomorphic line bundle over a complex manifold Y. The Hermitian fiber metric on L will be denoted by ϕ. In practice, ϕ is considered as a collection of local functions. Namely, let s be a local holomorphic trivializing section of L, then locally, $|s(z)|_{\phi}^{2}=e^{-\phi(z)}$. If α_k is a holomorphic section with values in L^k, then it may be locally written as $\alpha_k=f_k s^{\otimes k}$, where f_k is a local holomorphic function and the point-wise norm of α_k may be written as

$$|\alpha_k|_{k\phi}^{2}=|f_k|^{2}\,e^{-k\phi(z)}. \tag{2.1}$$

The canonical curvature two-form of L can be globally expressed as $\partial\overline{\partial}\phi$ and the normalized curvature form $i\partial\overline{\partial}\phi/2\pi=dd^c\phi$ (where $d^c:=i(-\partial+\overline{\partial})/4\pi$) represents the first Chern class $c_1(L)$ of L in the second real de Rham cohomology group of X [23]. A line bundle will be said to be *(semi-) positive* if there is some smooth metric ϕ on L with (semi-) positive curvature form (i.e. the matrix $(\frac{\partial^2\phi}{\partial z_i\partial\bar{z}_j})$ is (semi-) positive).

Let X be a smooth strictly pseudoconcave domain in Y. This means that there is a defining function ρ (i.e. $X=\{\rho\leq 0\}$ and $d\rho\neq 0$ on ∂X) such that the restriction of the Levi curvature form $\partial\overline{\partial}\rho$ to the maximal complex subbundle $T^{1,0}(\partial X)_x$ of the real tangentbundle of ∂X is negative (i.e. the Levi curvature of ∂X is negative). The degenerate case $X=Y$ is allowed in the previous definition of X and corresponds to the situation studied in [31, 11, 12] (when $dd^c\phi$ is *strictly* positive).

We will assume that Y is a projective manifold with a semi-positive line bundle L (which is positive at some point in Y). In case the curvature is positive on all of Y, the pair (Y,L) it usually called a *polarized manifold* in the literature. Fix a (possibly singular) Hermitian metric ϕ on L over Y whose curvature is a positive current [21].[1] The domain X in Y will be called a *polarized domain* if the metric ϕ on L is smooth on X and it will be called a *polarized domain with compatible curvatures* if any of the assumptions in section 2.2 below are satisfied. [2]

Fixing an Hermitian metric two-form ω on X (with associated volume form ω_n) the Hilbert space $\mathcal{H}(Y,L^k)_X$ is defined as the space $H^0(Y,L^k)$ with the norm obtained by restriction of the global norm on [23] to X :

$$\|\alpha_k\|_{k\phi}^{2}:=\|\alpha_k\|_{k\phi X}^{2}\,(=\int_X|f_k|^{2}\,e^{-k\phi(z)}\omega_n), \tag{2.2}$$

using a suggestive notation in the last equality (compare formula 2.1). If η is a form we will write $\eta_p:=\eta^p/p!$, so that the volume form on X may be written as ω_n. The induced volume form on ∂X will be denoted by $d\sigma$. If Z is a submanifold, then $[Z]$ will denote the corresponding current, i.e.

[1] the somewhat confusing terminology of positive currents actually means that $dd^c\phi$ is allowed to be a *semi*-positive form on the set where ϕ is smooth.

[2] Since a polarization usually refers to a *positive* line bundle L, the term semi-polarized would perhaps be more appropriate.

$([Z], \eta) := \int_Z \eta$ for any test form η. Moreover, given a real $(1,1)-$form η on Y we will denote by $\{\eta\}_y$ the corresponding (using the metric form ω) Hermitian linear operator on $T^{1,0}(Y)_y$ (or on some specified subbundle).

2.2. **Assumptions for "compatible curvatures".** At least one of the following three assumptions are assumed to be satisfied for X to be a *polarized domain with compatible curvatures* (compare [5]). The assumptions all have in common the condition that the slope function (see formula 1.5) is constant:

$$T \equiv C \text{ on } \partial X, \tag{2.3}$$

for some choice of the defining function ρ of ∂X.

Assumption 1. The defining function $-\rho$ of the pseudoconvex manifold $Y - X$ may be chosen to be smooth with $d\rho \neq 0$ and $dd^c(-\rho) > 0$ in $(Y - X) - Z$, where Z is either a point or an irreducible divisor in $Y - X$.[3] Moreover, on any regular sublevelset of ρ the slope function T in 1.5 (defined by replacing ∂X with the sublevelset of ρ) is constant, i.e.

$$T \text{ is a function of } \rho. \tag{2.4}$$

If Z is a point it is assumed that $dd^c(-\rho) > 0$ on all of $Y - X$. If Z is an irreducible divisor it is assumed that T is bounded from above on $Y - X$, that

$$\int_Z c_1(L)^{n-1} = 0. \tag{2.5}$$

and that

$$dd^c(-\rho) = [Z] + \beta, \tag{2.6}$$

in the sense of currents on $Y - X$, where β is a semi-positive smooth form.

Assumption 2. Suppose that $n \geq 2$ (the dimension of X) and that L is holomorphically trivial on $Y - X$. Then the fiber metric ϕ on L may be identified with a function on $Y - X$ and it as assumed that

$$\phi = -\rho$$

on $Y - X$. In this case the form μ in formula 1.5 is simply given by

$$\mu = (dd^c\phi)_{n-1} \wedge d^c\phi/n.$$

Assumption 3. Suppose that $n \geq 3$, that $Y - X$ is a Stein manifold and that

$$dd^c\phi = -f dd^c\rho \tag{2.7}$$

along the holomorphic tangentbundle of ∂X for some non-negative function f on ∂X.

[3] i.e. a (possibly singular) connected compact closed complex submanifold of codimension one in $Y - X$. Then the integration current $[Z]$ is well-defined. [23]

2.3. **General properties of Bergman kernels.** Let (ψ_i) be an orthonormal base for a given Hilbert space structure on the space $H^0(Y, L^k)$, which in this paper always will be the Hilbert space $\mathcal{H}_k(Y, L^k)_X$. The *Bergman kernel* of the Hilbert space $H^0(Y, L)$ is defined by

$$K^k(x, y) = \sum_i \psi_i(x) \otimes \overline{\psi_i(y)}.$$

Hence, $K^k(x, y)$ is a section of the pulled back line bundle $L^k \boxtimes \overline{L}^k$ over $Y \times Y$. For a fixed point y we identify $K^k_y(x) := K^k(x, y)$ with a section of the hermitian line bundle $L^k \otimes L^k_y$, where L_y denotes the line bundle over Y, whose constant fiber is the fiber of L over y, with the induced metric. The definition of K^k is made so that K^k satisfies the following reproducing property

$$\alpha(y) = (\alpha, K^k_y)_{k\phi} \tag{2.8}$$

[4]for any element α of $H^0(Y, L^k)$, which also shows that K^k is well-defined. In other words K^k is the integral kernel of the orthogonal projection onto $H^0(Y^k, L)$ in $L^2(Y, L^k)$. The restriction of K^k to the diagonal is a section of $L^k \otimes \overline{L}^k$ and we let $B^k(x) = \left|K^k(x, x)\right|_{k\phi} (= \left|K^k(x, x)\right| e^{-k\phi(x)})$ be its point wise norm:

$$B^k(x) = \sum_i |\psi_i(x)|^2_{k\phi}.$$

We will refer to $B^k(x)$ and $B^k 1_X \omega_n$ as the *Bergman function* and *Bergman measure of* $H^0(Y, L^k)$. Note that the Bergman measure only depends on the "restriction" of the metric ϕ to the domain X. The following extremal property holds:

$$B^k(x) = \sup |\alpha_k(x)|^2_{k\phi}, \tag{2.9}$$

where the supremum is taken over all L^2-normalized elements α_k of $H^0(Y, L^k)$. An element realizing the extremum, is called an *extremal at the point* x and is determined up to a complex constant of unit norm. Given such an extremal α the following basic relation holds [3]:

$$\left|K^k(x, y)\right|^2_{k\phi} = |\alpha_k(y)|^2_{k\phi} B^k(x) \tag{2.10}$$

3. Examples and "counter examples"

In this section we will consider various classes of polarized pseudoconcave domains. Some " counter examples" will also be presented, showing that the main results in this article may not hold if the assumptions in section 2.2 are relaxed.

[4]We are abusing notation here: the scalar product $(\cdot, \cdot)_{k\phi}$ on $H^0(Y, L^k)$ determines a pairing of K^k_y with any element of $H^0(Y, L^k)$, yielding an element of L^k_y.

3.1. **Domains in projective space and polynomials in $\mathbb{C}^n$.** Let Y be the $n-$dimensional projective space $\mathbb{P}^n$ and let L be the hyperplane line bundle $\mathcal{O}(1)$. Then $H^0(Y, L^k)$ is the space of homogeneous polynomials in $n+1$ homogeneous coordinates $Z_0, Z_1, ..Z_n$ [23]. The Fubini-Study metric ϕ_{FS} on $\mathcal{O}(1)$ may be suggestively written as $\phi_{FS}(Z) = \ln(|Z|^2)$ and the Fubini-Study metric ω_{FS} on $\mathbb{P}^n$ is the normalized curvature form $dd^c\phi_{FS}$, which is hence invariant under the standard action of $SU(n+1)$ on $\mathbb{P}^n$. We may identify $\mathbb{C}^n$ with the "affine piece" $\mathbb{P}^n - H_\infty$ where H_∞ is the "hyperplane at infinity" in $\mathbb{C}^n$ (defined as the set where $Z_0 = 0$). In terms of the standard trivialization of $\mathcal{O}(1)$ over $\mathbb{C}^n$, the space $H^0(Y, L^k)$ may be identified with the space of polynomials $p_k(\zeta)$ in $\mathbb{C}^n_\zeta$ of total degree at most k and the fiber metric ϕ may be identified with a (plurisubharmonic) function in $\mathbb{C}^n$.

The basic example of a pseudoconcave domain X is obtained as the complement in $\mathbb{P}^n$ of the unit-ball in $\mathbb{C}^n$ (i.e. we may take $\rho = -|\zeta|^2+1$ in a neighbourhood of ∂X). The norm 2.2 on the Hilbert space $\mathcal{H}_k(Y, L^k)_X$ may in this case be expressed as

$$\|p_k\|^2_{k\phi} := \int_{|z|>1} |p_k(\zeta)|^2 e^{-k\phi(\zeta)}(\omega_{FS})_n$$

Example 3.1. The canonical Fubini-Study metric on $\mathcal{O}(1)$ corresponds to the choice $\phi(\zeta) = \ln(1 + |\zeta|^2)$. In this case the (normalized) curvature of L is the Fubini-Study metric on $\mathbb{P}^n$ and the form μ in formula 1.5 is a multiple of the standard volume form on the $2n-1-$sphere.

Next, we will consider the basic example when ϕ has a singularity in $Y - X$.

Example 3.2. Let $\phi(\zeta) = \ln(|\zeta|^2)$. Then ϕ is smooth outside the origin. In particular it is smooth on X. Note that the nth exterior power of the curvature of ϕ vanishes outside the origin (i.e. the complex Monge-Ampere of ϕ vanishes there). Again, the form μ is a multiple of the standard volume form on the $2n-1-$sphere.

In the following section generalizations of the case when X is the complement of the unit-ball are considered (compare remark 3.5).

3.2. **Disc bundles.** Let Z_+ be a closed compact complex manifold of dimension $n-1$ and let (F, ϕ_F) and (G, ϕ_G) be Hermitian holomorphic line bundle over Z_+ with positive curvature. Then X is defined as the pseudoconcave domain obtained as the unit discbundle in the total space of F and Y as the $\mathbb{P}^1-$bundle over Z obtained by fiberwise "adding the point at infinity" to F, i.e. by adding a divisor Z_- at infinity. [5]Hence, locally

$$X = \{h = |w|^2 \exp(-\phi_F(z)) \leq 1\}$$

[5]i.e. Y is the fiber-wise projectivization of the bundle $F \oplus \underline{\mathbb{C}}$, where $\underline{\mathbb{C}}$ is the trivial line bundle over Z_+. The coordinate along $\underline{\mathbb{C}}$ determines a section of $\mathcal{O}_{\mathbb{P}(F\oplus\underline{\mathbb{C}})}(1)$ whose zero-set is Z_-.

(where z is a coordinate along Z and w is a coordinate along the fibers of F). We will assume that the "slope between $dd^c\phi_G$ and $dd^c\phi_F$":

$$S = \sup\{t \geq 0 : (dd^c\phi_G - tdd^c\phi_F)_z \geq 0\} \equiv S_0 \in \mathbb{Z}, \tag{3.1}$$

i.e. that S above is independent of the point z in Z and that $S \in \mathbb{Z}$. The line bundle L over Y is now defined as

$$L := \pi_F(G) \otimes [Z_-]^{S_0}$$

where $[Z_-]$ now denotes the *line bundle* over Y corresponding to the divisor Z_-(do that $c_1([Z_-])$ is represented by the *current* $[Z_-]$, using the notation introduced in section 2) Hence, L is isomorphic to $\pi_F^*(G)$ over the domain X. As will be seen below L satisfies the assumption 2.5 (with $Z = Z_-$). Moreover, the defining function $\rho := \ln h$ for the boundary of X satisfies the assumption 2.6 (with $\beta = dd^c\phi_F$). Let us now consider two different metrics on L :

Example 3.3. Let $\phi_L(z, w) := \pi_G^*\phi_G + \ln(1 + e^{S\rho})$ on $Y - Z_-$ (smoothly extended as a metric on L over a neighbourhood of Z_-). Then ϕ_L has positive curvature on $X - Z_-$ and $(dd^c\phi_G)^{n-1} = 0$ on Z_- precisely when assumption 3.1 holds. Indeed, a direct calculation (compare the proof of lemma 6.1) gives

$$dd^c\phi_L = \pi_F^*(dd^c\phi_G - s(\rho)dd^c\phi_F) + dt \wedge \pi_F^*d^c\rho,$$

where $s(\rho) = \frac{\partial}{\partial\rho}\ln(1 + e^{S\rho})$ is strictly increasing, mapping $[-\infty, \infty]$ to $[0, S]$. Note that the slope function T (see formula 1.5) becomes a function of $\rho : T(\rho) = S - s(\rho)$ in this case. Hence, all assumptions in 1 in section 2.2 are satisfied and (X, ϕ_L) is thus a polarized domain with compatible curvatures in (Y, L).

The following example may be obtained as a limit of variants of the previous one:

Example 3.4. Let $\phi_L(z, w) := \pi_F^*\phi_G(z)$ on X. Then ϕ_L extends to a singular metric on L over Y with positive curvature (in the sense of currents) by setting $\phi_L(z, w) := \phi_G(z) + \rho$ on $Y - X$. Indeed, ϕ_L may be obtained as the limit

$$\phi_L(z, w) := \lim_{k\to\infty}(\pi_G^*\phi_G + k^{-1}\ln(1 + e^{Sk\rho}))$$

Note that in this example $S = T$.

Remark 3.5. Setting $Z_+ = \mathbb{P}^{n-1}$ and $F = G = \mathcal{O}(1)$ gives the case considered in the previous section, i.e. when X is the complement of the unit-ball in $\mathbb{P}^n$. Indeed, the base Z_+ of the fibration above may be identified with the hyperplane at infinity in $\mathbb{C}^n$ and the fibers correspond to lines through the origin in $\mathbb{C}^n$. Moreover, Y corresponds to the blow-up $\widetilde{\mathbb{P}^n}$ at the origin of $\mathbb{P}^n$. A local isomorphism between Y and $\widetilde{\mathbb{P}^n}$ is obtained by setting $w = \zeta_n^{-1}$ and $z = (\zeta_1/\zeta_n, ..., \zeta_{n-1}/\zeta_n)$. Also note that Z_- corresponds to the exceptional divisor E over the origin and $L \approx \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$, where π is the blow-down map from $\widetilde{\mathbb{P}^n}$ to $\mathbb{P}^n$. Hence, the

space $H^0(Y, L^k)$ may be identified with $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^k)$ in this particular case.

3.3. **"Counter examples".**

3.3.1. *Constant vs. non-constant slope T.* Now assume that the base Z_+ is a product of complex curves: $Z_+ = Z_1 \times Z_2$, $\dim Z_i = 1$. All other objects are assumed to decompose accordingly: $F = F_1 \otimes F_2$ etc. Then the curvature form $dd^c\phi_F$ may be identified with a pair of functions (the eigenvalues of $dd^c\phi_F$) : $(\lambda_{F_1}(z_1), \lambda_{F_2}(z_2))$ and similarly for $dd^c\phi_G$. For concreteness, consider the case when

$$dd^c\phi_G \leftrightarrow (2,2), \;\; dd^c\phi_F \leftrightarrow (1+\epsilon_1(z_1), 2+\epsilon_2(z_2)),$$

where $|\epsilon_1(z_1)| < 1$, $|\epsilon_2(z_2)| < 2$ and the integral of $\epsilon_i(z_i)$ over Z_i vanishes. Hence, $(\lambda_{F_1}(z_1), \lambda_{F_2}(z_2))$ corresponds to a deformation (with positive curvature) of a metric on F with constant curvature. Now, if $\epsilon_1 \equiv 0$, then the slope function $T(z)$ is clearly *constant* $(= 1)$, as long as $|\epsilon_2(z_2)| \leq 1$. This means that the corresponding disc bundle X with the hermitian line bundle L defined as in example 3.4 is hence a polarized domain with compatible curvatures (since it satisfies the assumption 1 in section 2.2). But if $\epsilon_1 \neq 0$ somewhere and if $\epsilon_2(z_2) \leq \epsilon_1(z_1)+1$, then a short calculation gives

$$T(z) = 2/(1+\epsilon_1(z_1))$$

and T is hence *non-constant.* In fact, it can be shown that *the main results of this paper hold if and only if $T(z)$ is constant in these examples* (and certain more general examples) [7], which makes the assumption 2.3 quite natural.

Finally, consider the case when Z_+ is a complex curve, so that X is a domain in a complex surface (i.e. $n = 2$). Then $T(z) = dd^c\phi_G(z)/dd^c\phi_F(z)$. Now suppose that $Z_+ = \mathbb{P}^1$ and $F = G = \mathcal{O}(1)$ with $\phi_G(z) = \ln(1 + |z|^2)$ and $\phi_F(z) = \ln(a + |z|^2)$ (compare remark 3.5). Then X corresponds to the exterior of the "ellipse" $\{(\zeta_1, \zeta_2) : |\zeta_1|^2 + a\,|\zeta_2|^2 = 1\}$ in $\mathbb{C}^2$. Hence, $T(z)$ is constant precisely when $a = 1$, i.e. when X is the exterior of the unit-ball in $\mathbb{C}^2$. This example also shows the need to assume that $n > 2$ in assumption 3 in section 2.2. Indeed, when $n = 2$ the assumption 2.7 always holds. Similarly, the example also shows the need to assume extension properties of ρ and (L, ϕ), as in the assumptions 1 and 2 (at least when $n = 2$), since 2.3 always holds when $n = 2$ (for a suitable choice of ρ).

3.3.2. *Vanishing vs. non-vanishing of $c(L)^{n-1} \cdot [Z]$.* The following example illustrates the need for the condition 2.5 in the assumption 1 in section 2.2. Consider the situation in remark 3.5, but replace L with the line bundle $L = \pi^*\mathcal{O}_{\mathbb{P}^n}(2) \otimes [E]^{-1}$, Then L is a positive line bundle and satisfies all the assumptions in 1 in section 2.2, except 2.5. Indeed, L corresponds to $\pi_F(G) \otimes [Z_-]^{S_0-1}$ in section (with $G = \mathcal{O}_{\mathbb{P}^{n-1}}(2)$ and

$F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$). However, the assumption 2.5 fails, since

$$c(L)^{n-1} \cdot [E] = 0 + 0 + ... + 0 + [E]^n \neq 0$$

Note that L has a natural singular metric with curvature form $\pi^* c(\mathcal{O}(2))+$ E, where $c(\mathcal{O}(2))$ denotes the curvature form of a fixed smooth hermitian metric on $\mathcal{O}(2)$ with positive curvature, so that L is isomorphic to $\mathcal{O}(2)$ over X (as Hermitian holomorphic line bundles). Moreover, there is a strict inclusion

$$H^0(\widetilde{\mathbb{P}^n}, L^k) \hookrightarrow H^0(\widetilde{\mathbb{P}^n}, \pi^* \mathcal{O}_{\mathbb{P}^2}(2)^k), \tag{3.2}$$

where the image is the subspace in $H^0(\widetilde{\mathbb{P}^n}, \pi^* \mathcal{O}_{\mathbb{P}^2}(2)^k)$ of all sections vanishing along E to order k (i.e. the image may be identified with the subspace of all polynomials in $\mathbb{C}^n$ of total degree m, where $k < m \leq 2k$). It follows that the main results of this paper do not apply to $(\widetilde{\mathbb{P}^n}, L)$, since they imply that formula 4.1 holds, where the left hand side in the formula only depends on the restriction of the curvature of L to X. Indeed, the formula hold does hold for $(\widetilde{\mathbb{P}^n}, \pi^* \mathcal{O}_{\mathbb{P}^2}(2))$, since it satisfies all the assumptions in 1. Hence, by 3.2 it cannot hold for $(\widetilde{\mathbb{P}^n}, L)$.

4. Morse (in)equalities and model Bergman kernels

In this section we will mainly recall some point-wise estimates for the Bergman functions of the space $\mathcal{H}_k(Y, L^k)_X$ obtained in [2, 4]. Such point-wise estimates were referred to as *local holomorphic Morse inequalities* in [2] in the more general context of harmonic $(0, q)-$form with values in L^k.[6] After integration the latter estimates yield bounds for the asymptotic growth of the Dolbeault cohomology groups with values in L^k. The latter bounds were first obtained by Demailly [20] in the context of closed manifolds, who called them *holomorphic Morse inequalities* in analogy with the classical case of Morse inequalities for the De Rham cohomology groups of a real manifold (compare Witten's approach in [36]).

4.1. **Morse inequalities in the interior region.**

For a fixed k the interior region is defined by the inequality $\rho \leq -1/\ln k$.[7] In [4] it was shown that the Bergman function $B^k(x)$ may be estimated in terms of *model Bergman functions.* The model Bergman function B^0 associated to an interior point x is obtained by replacing the manifold X with $\mathbb{C}^n$ and the line bundle L with the constant curvature line bundle over $\mathbb{C}^n$ obtained by freezing the curvature of L at the point x. Since $\mathbb{C}^n$

[6] The case of holomorphic sections is considerably more elementary than the general case. The main difference is that there is no need for a special sequence of metrics on X as in [4] and that subelliptic estimates may be replaced by the submean property of holomorphic functions.

[7] in the following $\ln k$ could be replaced with any sequence R_k tending to infinity at the order $O(k^\epsilon)$ where ϵ is a sufficiently small positive number. Note that in [4] $R_k = R$ and the limit when first k and then R tend to infinity was considered. In this paper a slightly more precise control in the "boundary region" (see the appendix) will allow us to let R depend on k, hence simplifying the notation a bit.

is non-compact all sections are assumed to have finite L^2-norm. More concretely, one may always arrange so that locally around the fixed point x,

$$\phi(z) = \sum_{i=1}^{n} \lambda_i \left|z_i\right|^2 + ..., \;\; \omega(z) = \frac{i}{2}\sum_{i=1}^{n} dz_i \wedge \overline{dz_i} + ... \tag{4.1}$$

where the dots indicate lower order terms and the leading terms are called *model metrics.* Hence, the corresponding model L^2-norm on $\mathbb{C}^n$ is given by

$$\int_{\mathbb{C}^n} \left|\alpha(z)\right|^2 e^{-\sum_{i=1}^n \lambda_i |z_i|^2}, \tag{4.2}$$

integrating with respect to the Euclidean measure on $\mathbb{C}^n$.

Denote by F_k the holomorphic scaling map

$$F_k(z) = (z/k^{1/2})$$

and let $\alpha^{(k)}(z) := (F_k^*\alpha_k)(z)$. By the proof of theorem 1.1. in [2] (see also [3] for a simple argument based on the submean property of holomorphic functions) the following point-wise bound holds in the interior region:

$$\limsup_k k^{-n} \left|\alpha^{(k)}(z)\right|^2_{k\phi} / \left\|\alpha_k\right\|^2_{k\phi F_k(D_{\ln k})} \leq B^0(0), \tag{4.3}$$

where D_{lnk} denotes a polydisc of radius $\ln k$. In particular, by the extremal property 2.9 of $B^k(x)$:

$$\limsup_k k^{-n} B^{(k)}(z) \leq B^0(0) \tag{4.4}$$

Moreover, the model Bergman function is explicitly given by

$$B^0\omega_n = (dd^c\phi)_n \tag{4.5}$$

Hence, the full model Bergman kernel is given by

$$K^0 = \det(dd^c\phi_0)e^{\phi_0}, \tag{4.6}$$

using the suggestive notation $\phi_0 = \phi_0(z, z') = \sum_{i=1}^n \lambda_i \overline{z_i} z_i'$.

4.2. **Morse inequalities in the middle region.** The middle region is defined by the inequalities $-1/\ln k \leq \rho \leq -\ln k/k$. As shown in [4] (section 5.2)

$$\lim_k \int_{-1/\ln k \leq \rho \leq -\ln k/k.} k^{-n} B^k \omega_n = 0 \tag{4.7}$$

4.3. **Morse inequalities in the boundary region.** The boundary region is defined by the inequalities $-\ln k/k \leq \rho \leq 0$ and is diffeomorphic to the product $\partial X \times [-\ln k/k, 0]$. Fix a point σ in ∂X and take local holomorphic coordinates (z, w), where z is in $\mathbb{C}^{n-1}$ and $w = u + iv$. By an appropriate choice we may assume that the coordinates are orthonormal at 0 and that

$$\rho(z,w) = v + \sum_{i=1}^{n-1} \mu_i \left|z_i\right|^2 + O(\left|(z,w)\right|^3) =: \rho_0(z,w) + O(\left|(z,w)\right|^3). \tag{4.8}$$

In a suitable local holomorphic trivialization of L close to the boundary point σ, the fiber metric may be written as

$$\phi(z,w) = \sum_{i,j=1}^{n-1} \lambda_{ij} z_i \overline{z_j} + O(|w|)O(|z|) + O(|w|^2) + O(\left|(z,w)\right|^3),$$

where the leading terms are called the model fiber metric and denoted by ϕ_0. The model Bergman function B^0 and kernel K^0 associated to the fixed point σ are the ones obtained from the Hilbert space of all holomorphic functions α on the model domain X_0 (with defining function ρ_0) which are square integrable with respect to the model norm

$$\int_{X_0} \left|\alpha(z,w)\right|^2 e^{-\phi_0(z)},$$

integrating with respect to the Euclidean measure on $\mathbb{C}^n_{z,w}$.

Denote by F_k the holomorphic scaling map

$$F_k(z,w) = (z/k^{1/2}, w/k), \tag{4.9}$$

so that

$$X_k = F_k(D_{\ln k}) \bigcap X$$

is a sequence of decreasing neighborhoods of the boundary point σ, where $D_{\ln k}$ denotes the polydisc of radius $\ln k$ in $\mathbb{C}^n$. Note that

$$F_k^{-1}(X_k) \to X_0, \tag{4.10}$$

in a suitable sense. On $F_k^{-1}(X_k)$ we have the *scaled* fiber metric $F_k^* k\phi$ that tends to the model metric ϕ_0 on the model domain X_0, when k tends to infinity.

It follows from the proof of proposition 5.5 in [4] that

$$\limsup_k k^{-(n+1)} \left|\alpha^{(k)}(z,w)\right|^2_{k\phi} / \left\|\alpha_k\right\|^2_{k\phi X_k} \leq B^0(z,w), \tag{4.11}$$

where the estimate is uniform on $F_k^{-1}(X_k)$. Moreover, the left hand side above is uniformly bounded by a constant on any polydisc of fixed radius in $\mathbb{C}^n$ (even without intersecting with X) and on X_0 dominated by an L^1–function on “half-rays” (see the appendix). In particular,

$$(i) \quad \limsup_k k^{-(n+1)} B^{(k)}(z,w) \quad \leq \quad B^0(z,w) \tag{4.12}$$

The model Bergman function at the fixed point σ in ∂X is given by

$$B_{X_0}(z,w)\omega_n = \int_0^T e^{t\rho_0(z,w)} t(dd^c\phi_0 + tdd^c\rho_0)_{n-1} \wedge d^c\rho_0)dt, \tag{4.13}$$

where T is the slope function in formula 1.5. In particular, performing the fiber-integral (i.e the push forward) over ρ gives

$$\int_{\rho=-\infty}^0 B^0\omega_n = \mu, \tag{4.14}$$

where μ as the $2n-1$ form defined by formula 1.5. Similarly, the corresponding model Bergman kernel is given by the following integral formula

$$K^0 = \frac{1}{4\pi}\frac{1}{\pi}\int_0^T e^{t\rho_0+\phi_0} t\det(dd^c\phi_0 + tdd^c\rho_0)dt, \tag{4.15}$$

using a suggestive notation as in formula 4.6 above. Equivalently, we have the suggestive formula

$$K^0 = \frac{1}{4\pi}\frac{1}{\pi}\det(dd^c\rho_0)e^{\phi_0}P(\frac{\partial}{\partial\rho_0})\frac{\partial}{\partial\rho_0}(\frac{e^{T\rho_0}-1}{\rho_0}),$$

where P is the characteristic polynomial of the operator $\{dd^c\phi\}\{-dd^c\rho\}^{-1}$ where the operators act on $T^{1,0}(\partial X)_x$and $(P(\frac{\partial}{\partial\rho_0})$ denotes the corresponding differential operator with constant coefficients). Note that T is the minimal eigenvalue of $\{dd^c\phi\}\{-dd^c\rho\}^{-1}$.

4.4. **Morse equalities.** Integrating the Bergman function B^k over X gives, using the point-wise bounds in the different regions the following Morse inequalities for any line bundle L over X (compare [4])

$$\dim \mathcal{H}_k(Y)_X \leq k^n(\int_X (dd^c\phi)_n + \int_{\partial X}\mu) + o(k^n),$$

where the form μ is given by formula 1.5.

In the case when X is a polarized pseudoconcave domain (section 2.1) the previous inequality becomes an equality:

Proposition 4.1. *Consider the Hilbert space $\mathcal{H}_k(Y, L^k)_X$ associated to the polarized pseudoconcave domain X. Then*

$$\lim_k k^{-n}\dim \mathcal{H}_k(Y)_X = (\int_X (dd^c\phi)_n + \int_{\partial X}\mu) \tag{4.16}$$

Proof. The assumption 1 holds: If the extension of the fixed fiber metric ϕ to Y (that will also be denoted by ϕ in the sequel) is smooth, then the equality in 4.16 was obtained in [4] (section 7.1), but stated incorrectly there without any assumptions on the slope function T in formula 1.5). We will next repeat the argument in 4.16 and point out the corrections appearing in [5]. First, since L is a semi-positive line bundle the left hand side above is given by, using for example Demailly's strong Morse inequalities [20]:

$$\int_Y c^1(L)^n = \int_X c^1(L)^n + \int_{Y-X} c^1(L)^n.$$

Next, Stokes theorem is used to show that the integral over $Y-X$ coincides with the boundary integral in 4.16 (also using that $dd^c\phi$ is a smooth semi-positive form representing $c^1(L)$) (proposition 0.1 in [5]). Note that we may even assume that $dd^c\phi$ is strictly positive by adding ω/m to $dd^c\phi$, where ω is the curvature form of any fixed positive line bundle A on X and in the end letting the positive number m tend to zero in the integrals.

The case when the extension ϕ is singular may be reduced to the smooth case. Indeed, for a fixed m the positivity of $L^m \otimes A$ allows us to find a sequence Φ_j of smooth metrics on $L^m \otimes A$ with semi-positive curvature decreasing to Φ, where $dd^c\Phi = mdd^c\phi + \omega$ (using the regularization results of Demailly - see the appendix in [25] for a direct proof). This gives the Morse equality 4.16 with ϕ replaced by Φ_j. Finally, letting first j and then m tend to infinity then proves the proposition in the singular case, since the operator that maps ϕ to $(dd^c\phi)^p$ is continuous when applied to a locally bounded decreasing sequence of smooth plurisubharmonic functions, as first shown by Bedford-Taylor (see [26]).

The assumption 2 holds: in this case proposition 0.1 in [5] may be replaced by a direct application of Stokes theorem (compare [4], section 7.1).

The assumption 3 holds: under the assumption 2.7 the strong Morse inequalities obtained in [4] give 4.16 with $\dim \mathcal{H}_k(Y)_X (= H^0(Y, L^k))$ replaced by $H^0(X, L^k)$. But if $Y-X$ is a Stein manifold a standard extension argument then shows that $H^0(X, L^k) = H^0(Y, L^k)$, using that the Dolbeault cohomology group $H^{0,1}_{cpt}(M, F)$ for compactly supported forms is trivial for any line bundle F on a Stein manifold M of dimension at least three (setting $M = Y - X$ and $F = L^k$).[21] $\square$

The following simple generalization of lemma 2.2 in [3] will be used to convert the Morse equalities from the previous proposition to equalities for the *scaled* Bergman functions and kernels.

Lemma 4.2. *Assume that (M, ν) is a manifold with a smooth volume form ν and V an open subset of M. For each fixed parameter u defined in a bounded set U of Euclidean $\mathbb{R}^N$ let $\Phi_{k,u}$ be a diffeomorphism from V onto $\Phi_{k,u}(V)$ such that the Jacobian of $\Phi_{k,u}$ converges uniformly to 1, when k tends to infinity. Let f and f_k be functions in $L^1(V, \nu)$ with compact support in V and such that supp$f_k \subset \Phi_{k,u}(V)$ and such that the sequence f_k is dominated by a function in $L^1(V, \nu)$. Moreover, assume that*

$$(i) \lim_k \int_V f_k d\nu = \int_V f d\nu \quad \textit{and} \quad (ii) \limsup f_k(\Phi_{k,u}(x)) \leq f(x),$$

for almost all x in V. Then $\Phi_k^* f_k$ tends to f in $L^1((V, \nu) \times U)$. In particular, there is a subsequence $\Phi_{k_j}^* f_{k_j}$ such that $f_{k_j}(\Phi_{k_j,u}(x))$ converges to $f(x)$ for almost all pairs (x, u) in $V \times U$.

Proof. When $\Phi_{k,u}$ is the identity map (and $V = M$) the lemma was essentially obtained in [3]. But for completeness we recall the argument: By the assumption (i)

$$\limsup_k \int_V |f_k - f| \, d\nu = 2 \limsup_k \int_V \chi_{+,k}(f_k - f) d\nu,$$

where $\chi_{+,k}$ is the characteristic function of the set where $f_k - f$ is non-negative. The right hand side can be estimated by Fatou's lemma, which (by the dominated convergence theorem) is equivalent to the inequality

$$\limsup_k \int_X g_k d\nu \leq \int_X \limsup_k g_k d\nu,$$

if the sequence g_k is dominated by an L^1-function. Taking $g_k = \chi_k(f_k - f)$ and using the assumption (ii), finishes the proof of this special case.

Now, for a general map $\Phi_{k,u}$ let h_k be the function on $U \times V$ defined by

$$h_k(u, x) := (\Phi_{k,u}^* f_k)(x).$$

The assumption on the Jacobian of $\Phi_{k,u}$ combined with the assumption (i) shows (by the change of variables formula) that (i) also holds for h_k on $U \times V$. Indeed,

$$\int_V h_k d\nu = \int_V \Phi_{k,u}^* (f_k \Phi_{k,u}^{-1*} d\nu) = \int_{\Phi_{k,u}(V)} f_k \Phi_{k,u}^{-1*} d\nu$$

and by assumption $\Phi_{k,u}(V) \cap \operatorname{supp} f_k = V \cap \operatorname{supp} f_k$ and $\Phi_{k,u}^{-1*} d\nu \to d\nu$ uniformly, giving

$$\lim_k \int_V h_k d\nu = \lim_k \int_V f_k d\nu = \lim \int_V f d\nu$$

In particular, by Fubini's theorem, the corresponding equality holds over $U \times V$ too. Moreover, by assumption h_k is L^1-dominated. We may now apply the special case when $\Phi_{k,u}$ is the identity map to the sequence h_k on $U \times V$ and obtain that h_k tends to f for almost all pairs (x, u). $\square$

In section 6, we will also have use for the following

Lemma 4.3. *Assume that (M, ν) is a measure space. Let f and f_k be non-negative functions in $L^1(M, \nu)$ such that*

$$(i) \lim_k \int_M f_k d\nu = \int_M f d\nu \quad and \quad (ii) \liminf f_k(x) \geq f(x) \; a.e$$

Then f_k tends to f in $L^1(M, \nu)$. In particular, there is a subsequence f_{k_j}such that $f_{k_j}(x)$ converges to $f(x)$ for almost all x.

Proof. Reversing the roles of f_k and f in the beginning of the proof of the previous lemma, the assumption (i) gives

$$\limsup_k \int_X |f_k - f| \, d\nu = 2 \limsup_k \int_X \chi_{-,k}(f - f_k) d\nu,$$

where now $\chi_{-,k}$ is the set where $f_k < f$. Let $h_k := \chi_{-,k}(f - f_k)$ and note that by assumption (ii) we have that

$$\limsup_k h_k = 0$$

and h_k is, by its definition, dominated by the L^1-function f. Hence, Fatou's lemma again finishes the proof of the lemma. □

5. Bergman kernel asymptotics

5.1. **Convergence as a current.**

Theorem 5.1. *Let B^k be the Bergman function for the Hilbert space $\mathcal{H}_k(Y,,L^k)_X$ associated to the polarized pseudoconcave domain X (section 2.1). Then*

$$k^{-n}B^k 1_X \omega_n \to 1_X(dd^c\phi)_n + [\partial X] \wedge \mu$$

as measures on Y in the weak-topology, where μ is the $2n-1$ form 1.5 on ∂X.*

Proof. For simplicity we will assume that the restriction of ρ to the ray close to the boundary where z and the real part of w vanish, coincides with the restriction of v (the assumption may be removed as in the proof of proposition 5.5 in [4]). Let $B_X^k(x) := k^{-n}B^k(x)$ when x is in the interior region, i.e. $\rho(x) \leq -1/\ln k$ and 0 otherwise and let

$$B_{\partial X}^k(\sigma) = k^{-n} \int_{-\ln k/k}^{0} B^k(\sigma, \rho) d\rho \tag{5.1}$$

By formula 4.7

$$\lim_k k^{-n} B^k \omega_n = \lim_k B_X^k \omega_n + [\partial X] \wedge (\lim_k B_{\partial X}^k(\sigma)) d\sigma$$

as currents. In particular, proposition 4.1 gives

$$\lim_k \int_X B_X^k \omega_n + \lim_k \int_{\partial X} B_{\partial X}^k(\sigma) d\sigma = \int_X (dd^c\phi)_n + \int_{\partial X} \mu.$$

Hence, the inequalities 5.7 and 4.12 show that the first and second term on the left hand side in the previous formula is equal to the first term and the second term, respectively, in the right hands side. Finally, lemma 4.2 applied to the spaces X and ∂X gives

$$\begin{array}{lrcl} (i) & \lim_k B_{X,R}^k \omega_n & = & (dd^c\phi)_n \text{ a.e on } X \\ (ii) & (\lim_k B_{\partial X}^k(\sigma) d\sigma) & = & \mu \text{ a.e on } \partial X. \end{array} \tag{5.2}$$

This proves the proposition. □

Now we can prove the convergence as a current of the whole Bergman kernel, stated as theorem 1.1 in the introduction.

Proof of theorem 1.1. First observe that
(5.3)

$$\lim_k k^{-n} \int_{X\times X} f(x,y) \left|K^k(x,y)\right|^2_{k\phi} \omega_n(x)\wedge\omega_n(y) = \lim_k I_{1,k} + \lim_k I_{2,k},$$

where $I_{1,k}$ and $I_{2,k}$ are the integrals obtained by restricting the integration to the set of all (x,y) such that $\rho(x),\rho(y) \leq -1/\ln k$ and $\rho(x),\rho(y) \geq -\ln k/k^1$, respectively. Indeed, if A_k denotes the middle region, i.e. the set of all x such that $-1/\ln k \leq \rho(x) \leq -\ln k/k$, the absolute value of the difference between the integrals in left hand side and the right hand side in 5.3 may be estimated by

$$C\int_{A_k}(\int_X \left|K^k(x,y)\right|^2_{k\phi} \omega_n(y))\omega_n(x) \leq C\int_{A_k} B^k(x)\omega_n(x)$$

using the reproducing property 2.8 of $K^k_x(y)$ applied to $\alpha_k = K^k_x(y)$ in the last step. By 4.7 the latter integral tends to zero, when k tends to infinity.

Proof. Since the proof that

$$\lim_k I_{1,k} = (\int_X f(x,x)(dd^c\phi)_n \tag{5.4}$$

is completely analogous to the case when X is closed (theorem 2.4 in [3]), we will just sketch it here. Take a sequence of sections α_k, where α_k is a normalized extremal at the interior point x. Combining the inequality 4.4 with the equality (i) in formula 5.2 shows that, unless x lies in a set of measure zero, there is a subsequence of α_k such that

$$\lim_k \|\alpha_k\|^2_{k\phi,F_k(D_{\ln k})} = 1, \tag{5.5}$$

restricting the norms to $F_k(D_{\ln k})$, the polydisc of radius $\ln k/k$ centered at x. Using the identity 2.10 the integral over y in $I_{1,k,R}$ (for a fixed point x) equals

$$k^{-n}B^k(x)\int_{\rho(y)\leq -1/\ln k} f(x,y)\left|\alpha_k(y)\right|^2_{k\phi}\omega_n(y).$$

Since by 5.5 the function $\left|\alpha_k(y)\right|^2_{k\phi}$ converges to the Dirac measure at x in the weak*-topology, formula 5.5 then proves 5.4.

Similarly to prove

$$\lim_k I_{2,k} = (\int_{\partial X} f(x,x)\mu \tag{5.6}$$

first note that in the limit f may clearly be replaced by its restriction to $(\partial X)^2$. Replace x in the previous argument with $x_k = (\sigma_x, v/k)$ and observe that

$$\lim_k \|\alpha_k\|^2_{k\phi F_k(\Delta_{\ln k})} = 1, \tag{5.7}$$

restricting the norms to the polydisc of radius $\ln k$ scaled by the map F_k (in formula 4.9). To see this note that combining (ii) in formula 5.2

with the inequality 4.12 gives, using lemma 4.2 applied to the infinite ray $\{\sigma\} \times [0, \infty[$,

$$\lim_k k^{-(n+1)} B^k(\sigma, v/k) = B^0(0, v)$$

for almost all (σ, v). The inequality 4.11 then gives 5.7 as in the interior case. The integral over y in the definition of $I_{2,k}$ (for a fixed point x) now equals

$$k^{-n} B^k(\sigma_x, \rho_x) \int_{\rho(y) \geq -\ln k/k^1} f(\sigma_x, \sigma_y) \left|\alpha_k(\sigma_y, \rho_y)\right|^2_{k\phi} \omega_n(y).$$

By 5.7 α_k is peaked around $\sigma_y = \sigma_x$ showing that 5.6 is the limit of

$$k^{-n} B^k(\sigma_x, \rho_x) f(\sigma_x, \sigma_x) \int \left|\alpha_k(\sigma_y, \rho_y)\right|^2_{k\phi} \omega_n(y).$$

Since the integral of $\left|\alpha_k(\sigma_y, \rho_y)\right|^2$ is equal to one in the limit (by 5.7) formula 5.2 finally proves 5.4. □

5.2. **Scaling asymptotics.** In this section scaling asymptotics for the Bergman kernels in the interior of X and at the boundary of X will be obtained. The scalings are expressed in terms of the local coordinates introduced in section 4.1 and 4.3, respectively. In the interior case we will use the notation $B^{(k)}(z) = B^k(z/k^{1/2})$ and $K^{(k)}(z, w) = K_z^{(k)}(w) = K(z/k^{1/2}, w/k^{1/2})$ and similarly in the boundary case, using the scaling map 4.9 in the latter case. Note that we have suppressed the dependence on the fixed center (which is the point x in the interiour and the point σ at the boundary).

Lemma 5.2. . *Let ϕ be any smooth Hermitian metric on the line bundle L over Y. Then the scaled Bergman kernels around each fixed interior point x satisfy*

$$(i) \limsup_k \left\|k^{-n} K_z^{(k)}\right\|^2_{k\phi_0} \leq \left\|K_z^0\right\|^2_{\phi_0} (= B^0(z)),$$

in terms of the model norms (restricted to a polydisc of radius ln k in the left hand side). Moreover, the left hand side is uniformly bounded by a constant independent of z. Similarly, for each fixed boundary point σ

$$(ii) \limsup_k \left\|k^{-(n+1)} K_{z,w}^{(k)}\right\|^2_{k\phi_0} \leq \left\|K_{z,w}^0\right\|^2_{\phi_0} (= B^0(z, w)),$$

in terms of the model norms (restricted to a polydisc of radius ln k in the left hand side).

Proof. By formula 2.10

$$\left\|k^{-n} K_z^{(k)}\right\|^2_{k\phi} = k^{-n} B^{(k)}(z) (k^{-n} \left\|\alpha^{(k)}\right\|^2_{k\phi}), \tag{5.8}$$

where α_k is an extremal at the point $x_k = z/k^{1/2}$ with global norm equal to one. Hence,

$$\limsup_k \left\|k^{-n} K_z^{(k)}\right\|^2_{k\phi_0} \leq \limsup_k k^{-n} B^{(k)}(z) \leq B^0(z),$$

where we have used the Morse inequality 4.4 in the final step. By the reproducing property of the Bergman kernel (or the analog of formula 5.8) in the model case this proves (i). The proof of (ii) follows along the same lines, now using the Morse inequities 4.12. $\square$

Lemma 5.3. *Let B^k be the Bergman function for the Hilbert space $\mathcal{H}_k(Y, L^k)_X$ associated to the polarized pseudoconcave domain X (section 2.1). Then B^k has a subsequence B^k_j such that for almost any point x in the interior of X (i.e. $x \in X - E$, where E has measure zero) the following scaling asymptotics centered at x hold:*

$$(i)\, k_j^{-n} B^{(k_j)}(z) = B^0(z) \tag{5.9}$$

for almost all z. Similarly, for almost any fixed boundary point σ (i.e. $\sigma a \in \partial X - F$, where F has measure zero in ∂X)

$$(ii)\, \lim_k k_j^{-(n+1)} B^{(k_j)}(z, w) = B^0(z, w)$$

for almost all (z, w).

Proof. First consider the interior case. Taking $\Phi_k = Id$ (i.e. the identity map) in lemma 4.2 and using the Morse inequality 4.4 applied to the center (i.e. to $k^{-n}B^{(k)}(0) = k^{-n}B^k(x)$) proves 5.9 when $z = 0$, i.e. that

$$\lim_k k^{-n} B^k(x) = B_0(x) \tag{5.10}$$

a.e. on X. Now fix a point x_0 in X and a coordinate neighbourhood V centered at x_0 that we identify with a subset of $\mathbb{C}^n$. Let U be a ball of fixed radius centered at the origin in $\mathbb{C}^n$. On V we may write

$$B^{(k)}(A(z)u) = B^k(\Phi_{k,u}(z)) := B^k(z + A(z)u/k^{1/2}),$$

where $A(z)$ is a matrix-valued function and where the center of the scaling is z.[8] On U the matrix $A(z)$ may even be chosen to depend smoothly on z. Note that the norm of the Jacobian of $\Phi_{k,u} - Id$ is bounded by a constant times $1/k^{n/2}$. Now take a smooth function χ supported on U such that $\chi = 1$ on some neighbourhood $V(x_0)$ of the fixed point x_0. Let $f_k := \chi k^{-n} B^k$ and $f := \chi B_0$ on V. By 5.10 (and dominated convergence) we have

$$\lim_k \int_V f_k \omega_n = \int_V f \omega_n$$

Applying lemma 4.2 to f_k with $(M, \nu) = (X, \omega_n)$ and V and U as above and using the Morse inequality 4.12 proves that f_k tends to f for almost all (x, u) in $V \times U$. In particular, $k^{-n}B^k(\Phi_{k,u}(x))$ tends to $B_0(x)$ for almost all (x, u) in $V(x_0) \times U$. By Fubini's theorem this means that for almost all fixed x in $V(x_0)$ we have that $k^{-n}B^k(\Phi_{k,u}(x))$ tends to $B_0(x)$

[8] recall that the definition of $B^{(k)}$ involves a choice of coordinates, that are orthonormal at z, corresponding to multiplying $A(z)$ by a unitary matrix. But it is clearly enough to obtained the scaling asymptotics for some choice of $A(z)$.

for almost all u in U. But since x_0 was arbitrary this proves the interior case.

To prove the boundary case, first note that as in the proof of theorem 5.1,

$$\lim_k \int_{\partial X \times [-\ln k, 0]} B^k(\sigma, v/k) dv d\sigma) = \int_{\partial X \times [-\ln k, 0]} B^0(\sigma, v) dv d\sigma).$$

The proof now follows along the lines of the interior case, using the Morse inequality 4.12 for B^k. □

We now turn to the proof of the scaling convergence of the Bergman kernel stated as theorem 1.2 in the introduction.

Proof of theorem 1.2. We first consider the interior case. Fix a point x in $X - E$ where E is the set of measure zero where the convergence in lemma 5.3 fails. By the uniform bound in lemma 5.2, the sequence $k_j^{-n} K_z^{(k_j)}$ extended by zero converges (after choosing a subsequence) weakly to an element β_z in the model space. Moreover, since $k_j^{-n} K_z^{(k_j)}$ is a holomorphic function the L^2- bounds may, using Cauchy estimates, be converted to $\mathcal{C}^\infty-$convergence on any given compact set. In particular,

$$\lim_j k_j^{-n} K_z^{(k_j)}(z) = \beta_z(z).$$

By lemma 5.3 (which is equivalent to the corresponding asymptotic identity for $K_z^{(k)}(z)$) this means that

$$\beta_z(z) = K_z^0(z) \tag{5.11}$$

under the assumptions of the theorem. The inequality (i) in lemma 5.2 combined with the extremal characterization 2.9 of the Bergman function B_k then forces

$$\beta_z(w) = c_z K_z^0(w) \tag{5.12}$$

for each w, where c_z is of unit norm for each fixed z. Combining 5.11 and 5.12 when $z = w$ shows that $c_z = 1$. All in all we deduce that, for each fixed z,

$$\lim_j k_j^{-n} K_z^{(k_j)} = K_z^0$$

uniformly on any given compact set. The limit has been established for a certain subsequence of $k_j^{-n} K^{(k_j)}$, but in fact it implies point-wise convergence of the sequence $k_j^{-n} K^{(k_j)}$ itself since the limiting function is independent of the subsequence of $k_j^{-n} K^{(k_j)}$. Moreover, by the uniform bound (i) in lemma 5.2

$$\left\| K^{(k_j)} - K^0 \right\| \leq C$$

in L^2 for each fixed compact set in $\mathbb{C}^n \times \mathbb{C}^n$. Since the sequence $(K^{(k_j)} - K^0)$ is holomorphic on $\overline{\mathbb{C}^n} \times \mathbb{C}^n$ Cauchy estimates finally may be used again to convert the L^2- convergence on $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathcal{C}^\infty-$convergence on

any fixed compact set of $\mathbb{C}^n \times \mathbb{C}^n$, proving the theorem in the interior case.

The boundary case follows along the same lines, now using (ii) in lemma 5.2 and 5.3.

6. Bergman metric asymptotics

Denote by Ω_k the following global $(1,1)-$current on Y :

$$\Omega_k := dd^c(\ln K^k(y,y))(= dd^c(\sum_i |\psi_i(y)|^2))$$

where $K^k(y,y)$ is the Bergman kernel of the Hilbert space $\mathcal{H}_k(Y, L^k)_X$ (with orthonormal basis (ψ_i)), restricted to the diagonal and identified with a section of $L \otimes \overline{L}$ over Y. Equivalently, Ω_k is the pull-back of the Fubini-Study metric ω_{FS} on $\mathbb{P}^N(= \mathbb{P}\mathcal{H}_k(Y)_X)$ (compare section 3) under the Kodaira map

$$Y \to \mathbb{P}\mathcal{H}_k(Y)_X, \;\; y \mapsto (\Psi_1(y) : \Psi_2(y) ... : \Psi_N(y)) \; ,$$

where (Ψ_i) is an orthonormal base for $\mathcal{H}_k(Y)_X$, i.e Ω_k is the (normalized) curvature of the metric $\ln K^k(y,y)$ on L, which is the pull-back of the Fubini-Study metric on the hyper plane line bundle $\mathcal{O}(1)$ over $\mathbb{P}^N(= \mathbb{P}\mathcal{H}_k(Y)_X)$. We will call Ω_k the *kth Bergman metric on Y induced by the polarized domain X.*

Now fix a point σ in ∂X and recall that B^0 and K^0 denote the corresponding model Bergman function and kernel, respectively, on $\mathbb{C}^n_y$ defined in section 4.3.

Lemma 6.1. *Let T be defined as in formula 1.5. Then*

$$dd^c(\ln K^0(y,y)) = td(d^c\rho_0) + dd^c\phi_0 + dt \wedge d^c\rho_0$$

where $t = \frac{\partial}{\partial\rho_0}\ln B^0(\rho_0)$ is strictly increasing, mapping $[-\infty, \infty]$ to $[0,T]$.

Proof. Consider $\psi(\rho_0) = \ln B^0(\rho_0)$ as a function on $\mathbb{C}^n$. Then

$$dd^c\psi = d(\frac{\partial\psi}{\partial\rho_0}d^c\rho_0) = td(d^c\rho_0) + dt \wedge d^c\rho_0,$$

where we have used Leibniz rule in the last step and the definition of t above. To prove the last statement above note that ψ is of the general form

$$\psi(y) = \ln \int_K e^{\langle y,t \rangle} d\nu(t),$$

where y and t are vectors in Euclidean $\mathbb{R}^N$ and $d\nu(t)$ is a finite measure supported on a compact set K. Hence, ψ is convex and it follows from well-known convex analysis [24] that the gradient of ψ maps $\mathbb{R}^N$ bijectively onto the interior of K. When $N = 1$ and $K = [a,b]$ it also follows that $-\infty$ and ∞ are mapped to a and b, respectively. $\square$

Next, we will prove theorem 1.3 about the convergence of the Bergman metric.

Proof of theorem 1.3. Let us first prove the weak convergence of the (normalized) sequence of Bergman volume forms $(dd^c(k^{-1}\ln K^k(y,y))_n$. Since the mass of this sequence of measures is bounded (compare 6.5 below), it is by weak compactness enough to show that any subsequence has another subsequence that converges to the expected limit. Now given a first choice of subsequence it has itself a subsequence such that the scaling convergence in theorem 1.2 holds. Hence, it will be enough to show weak convergence for the latter subsequence and to simplify the notation we will assume that it is indexed by k in the following.

Let $G^k(y) := (dd^c(k^{-1}\ln K^k(y,y))_n/\omega_n$ and

$$G^k_X(y) := 1_{\{\rho\leq -1/\ln k\}}G^k(y), \;\; G^k_{\partial X}(y) := \int_{-\ln k/k}^{\ln k/k} G^k(\sigma,\rho)d\rho$$

where σ denotes a point in ∂X as in the proof of theorem 5.1. To prove the weak convergence, i.e. that for any smooth "test function" f :

$$\lim_k \int G^k f\omega_n = \int_{X(0)} f(dd^c\phi)_n + \int_{\partial X} f\mu \tag{6.1}$$

it is clearly enough, by decomposing the previous integral into different regions, to prove the following

Claim 6.2. The following holds:

$$\begin{array}{lrcll}
(a) & G^k_X & \to & 1_{X(0)}(dd^c\phi)_n/\omega_n & \text{in } L^1(X,\omega_n) \\
(b) & G^k_{\partial X} & \to & \mu/d\sigma & \text{in } L^1(\partial X, d\sigma) \\
(c) & \int_{-1/\ln k\leq\rho\leq -\ln k/k} G^k\omega_n & \to & 0 & \\
(d) & \int_{R_k/k\leq\rho} G^k\omega_n & \to & 0 &
\end{array}$$

Proof. First observe that the following holds:

$$(a'):\;\; G^k_X(x) \to (1_{X(0)}(dd^c\phi)_n/\omega_n)(x) \text{ a.e. on } (X(0),\omega_n)$$

Indeed, consider a fixed point x in the interior region and take local coordinates z centered and orthonormal at x. Write $z = \zeta/k$. Then the chain rule gives

$$\partial\overline{\partial}(k^{-1}\ln K^k) = \sum_{i,j} \frac{\partial^2 \ln K^{(k)}(\zeta))}{\partial\zeta_i\partial\bar{\zeta}_j} dz_i\wedge d\bar{z}_j \tag{6.2}$$

Note that for each fixed k we may replace $K^{(k)}(\zeta)$ in the formula above by $k^{-n}K^{(k)}(\zeta)$, since $\partial^2(\ln k^{-n}) = 0$. Now, by the scaling convergence (i) in theorem 1.2 evaluating 6.2 at 0 shows that at almost any fixed point x :

$$\lim_k(\partial\overline{\partial}(k^{-1}\ln K^k))_n = (\sum_i \lambda_{x,i}dz_i\wedge d\bar{z}_i)_n = (\partial\overline{\partial}\phi)_n \;.$$

Next, we will show

$$(b'):\;\; \liminf_k G^k_{\partial X}(\sigma) \geq (\mu/d\sigma)(\sigma) \text{ a.e. on } (\partial X, d\sigma)$$

To this end fix a point σ in $\partial X - E$, where E is the set of measure zero where the scaling convergence (ii) in theorem 1.2 fails. Take local holomorphic coordinates (z, w) as in section 4.3. Recall that z is in $\mathbb{C}^{n-1}$ and $w = u + iv$. After a change of variables, $G^k_{\partial X}$ may be written in the following way:

$$G^k_{\partial X}(\sigma) = \int_{-\ln k}^{\ln k} k^{-1} G^k(0, iv'/k)dv' \tag{6.3}$$

(we are making the same simplifying assumptions on the fixed ray close to the boundary as in the beginning of the proof of theorem 5.1). Introduce the scaled coordinates $\zeta = (zk^{1/2}, wk)$. Then the integrand above may be written as

$$k^{-1}G^k(0, iv'/k) = \det(\frac{\partial^2 \ln K^{(k)}}{\partial\zeta_i \partial\bar{\zeta}_j})(0, iv'). \tag{6.4}$$

Indeed, from the definition of G^k we have that the left hand side in the formula above may be written as

$$\det(\frac{\partial^2 \ln K^{(k)}}{\partial\zeta_i \partial\bar{\zeta}_j})(\zeta) \cdot (k^{-(n+1)}(d\zeta_1 \wedge d\bar{\zeta}_1 \cdots)/(dz_1 \wedge d\bar{z}_1 \cdots dw \wedge d\bar{w}))$$

By the definition of the scaled coordinates ζ the second factor is equal to one and evaluating the expression at $\zeta = (0, iv')$ then proves 6.4.

As in the interior case above, we may now multiply $K^{(k)}$ in 6.4 by a factor $k^{-(n+1)}$ and apply theorem 1.2 (ii) combined with lemma 6.1 to obtain

$$\lim_k k^{-1}G^k(0, iv'/k)dv' = \det(dd^c\phi + tdd^c\rho)dt,$$

where t is a function of v'. Fatou's lemma combined with the change of variables $t = t(v')$ in the integral 6.3 then proves (b').

Now observe that 6.1 holds when $f = 1$. Indeed, since $dd^c(k^{-1}\ln K^k)$ represents the first Chern class of L over Y this follows as in the proof of proposition 4.1. In particular, splitting the integral gives the following *upper* bounds on the quantities in the claim 6.2:

$$\begin{gathered}\int_{X(0)} (dd^c\phi)_n + \int_{\partial X} \mu \geq \\ \limsup_k \int G^k_X \omega_n + \liminf_k \int_{\partial X} G^k_{\partial X} d\sigma + \\ \liminf_k (\int_{-1/\ln k_k \leq \rho \leq -\ln k_k/k} G^k\omega_n + \int_{\ln k_k/k \leq \rho} G^k\omega_n)\end{gathered} \tag{6.5}$$

Moreover, the previous bound clearly also holds whith $\limsup$ in front of any of the other two integrals (as long as the remaing integrals have $\liminf$ in front of them) . But then the lower bounds in (a') and (b') above combined with Fatou's lemma force

$$\lim_k \int G^k_X \omega_n = \int_{X(0)} (dd^c\phi)_n, \ \lim_k \int_{\partial X} G^k_{\partial X} d\sigma = \int_{\partial X} \mu. \tag{6.6}$$

Hence, (a) and (b) in the claim 6.2 follow from combininig these two limits with (a') and (b') above, using the integration lemma 4.3. Finally, combing 6.5 and 6.6 proves (c) and (d).

The statements (i) and (ii) of the theorem follow directly from theorem 1.2 combined with lemma 6.1. $\square$

7. Equilibrium measures

In this section we will take X to be *any* given compact set in Y and ϕ any given metric on $L \to Y$ which is continuous on X (only the "restriction" of ϕ to X will be relevant in the sequal). We will, for simplicity, assume that L is semi-positive, i.e that it admits some smooth metric with semi-positive curvature (see [6, 9] for the setup in the general case). Comparing with the previous sections we will say that X (or rather the pair (X, ϕ)) is a *(semi-)polarized set* if ϕ is smooth on X with (semi-)positive curvature form on all of X.

7.1. **Equilibrium metrics.** To a general pair (X, ϕ) we may associate the following *equilibrium metric* on $L \to Y$:

$$\phi_e(y) = \sup \left\{ \widetilde{\phi}(y) : \widetilde{\phi} \in \mathcal{L}_{(X,L)}, \, \widetilde{\phi} \leq \phi \text{ on } X \right\}. \tag{7.1}$$

where $\mathcal{L}_{(X,L)}$ is the class consisting of all (possibly singular) metrics on L with positive curvature current. Then the upper semi-continuous (usc) regularization ϕ_e^* is in $\mathcal{L}_{(X,L)}$ and is locally bounded [25]. In particular, the Monge-Ampere measure $(dd^c\phi_e^*)^n/n!$ is a well-defined positive measure by the classical work of Bedford-Taylor [26], which is supported on X and called the *equilibrium measure* associated to (X, ϕ). It was recently introduced in the more general global setting of quasiplurisubharmonic functions by Guedj-Zeriahi [25].

7.2. **Regularity.** In case (X, ϕ) is a semi-polarized domain one has that $\phi_e = \phi$ on the interiour of X [25], hence the non-trivial contribution to the equilibrium measure then comes from the boundary of X. The situation when X is all of Y, but ϕ is any (typically non-positively curved) smooth metric on L is studied in [6]. In the latter case it follows directly that ϕ_e is usc, i.e. $\phi_e^* = \phi_e$. In the general case the latter property holds precisely when $\phi_e^* = \phi_e$ on X and we will then say that (X, ϕ) is *regular,* using classical terminology[26]. In fact, $\phi_e^* = \phi_e$ on X precisely when ϕ_e is continuous on all of Y. As we will not consider regularity issues we refer the interested reader to [9] for a recent account, based on the classical work by Siciak and others. For example, when X is a domain with smooth boundary (X, ϕ) is always regular (as long as ϕ is continuous).

Remark 7.1. Consider the case when X is a polarized domain in Y with smooth boundary, so that $\phi_e^* = \phi_e$, which is equal to ϕ on X. Then $(dd^c\phi_e)^n/n! = 0$ on the complement of X [25] and by the "domination principle" [9] ϕ_e may then be caracterized as the unique extension of ϕ

from X to all of Y which solves the Dirichlet problem for the Monge-Ampere operator on $Y - X$. Using this it should be possibly to obtain the convergence of the Monge-Ampere operators in theorem 1.4, in the special case when X is polarized pseudoconcave domain with compatible curvatures, from theorem 1.3 (compare the approach in [29]).

7.3. **Bernstein-Markov measures and general Bergman kernels.** Extending classical terminology (compare [16, 9]) a measure ν is said to satisfy the *Bernstein-Markov property with respect to* (X, ϕ) if for any positive number ϵ there is a constant C_ϵ such that the following inequality holds for all positive integers k :

$$\sup_{x \in X} |\alpha_k|^2_{k\phi}(x) \leq C_\epsilon e^{k\epsilon} \int_Y |\alpha_k|^2_{k\phi} \nu \tag{7.2}$$

for any element α_k of $H^0(Y, L^k)$. Given such a measure ν one obtains a Hilbert space structure on $H^0(Y, L^k)$ by replacing the measure $1_X\omega_n$ in formula 2.2 with ν. We will denote by K^k the corresponding Bergman kernel, which hence depends on (ν, ϕ) and by $B^k\nu$ the corresponding Bergman measure.

For example, if X is a smooth domain and $\nu = 1_X\omega_n$, where ω_n is a smooth volume form on Y then ν has the Bernstein-Markov property with respect to (X, ϕ) (compare [9] where this is proved by adapting classical arguments of Siciak and others).

7.4. **The proof of theorem 1.4.** In the proof of the theorem 1.4 we will make use of the following well-known extension lemma, which follows from the Ohsawa-Takegoshi theorem (compare [10]):

Lemma 7.2. *Let (L, ϕ') be a (singular) Hermitian line bundle such that ϕ' has positive curvature form and let (A, ϕ_A) be an ample line bundle with a smooth (but not necesserly positively curved) metric ϕ_A. Then the following holds after replacing A by a sufficiently high tensor power: for any point y where $\phi' \neq \infty$, there is an element α in $H^0(Y, L^k \otimes A)$ such that*

$$|\alpha_k(y)|_{k\phi'} = 1, \quad \|\alpha_k\|_{Y,k\phi'} \leq C. \tag{7.3}$$

The constant C is independent of the point y and the power k.

The next lemma is used to reduce the general case to the case when L is an ample line bundle.

Lemma 7.3. *Let (A, ϕ_A) be a positive smooth Hermitian line bundle and let $\phi_m := m\phi + \phi_A$ be the induced metric on $L^m \otimes A$. Then*

$$((m\phi + \phi_A)_e - \phi_A)/m \to \phi_e \tag{7.4}$$

is a decreasing limit. Moreover, applying the Monge-Ampere operator to both sides gives a weakly convergent sequence of measures on Y.

Proof. To simplify the notation we set $\epsilon = 1/m$ and interpretate $\phi + \epsilon\phi_A$ as a $\mathbb{Q}-$metric on the $\mathbb{Q}$- line bundle $A^\epsilon \otimes L$ (these can be defined either in analogy with $\mathbb{Q}-$divisors [27], or in terms of quasiplurisubharmoic functions as in remark 7.7). Then $\Phi_\epsilon := (\phi + \epsilon\phi_A)_e - \phi_A$ is the sequence of metrics on L given by the left hand side in 7.4. To see that Φ_ϵ decreases as ϵdecreases to 0 it is clearly equivalent to prove

$$\epsilon \geq \epsilon' \Rightarrow (\phi + \epsilon\phi_A)_e \geq (\phi + \epsilon'\phi_A)_e + (\epsilon - \epsilon')\phi_A.$$

But this follows since the right hand side is a contender for the sup defining $(\phi + \epsilon\phi_A)_e$ (using that $dd^c\phi_A \geq 0$). As a consequence $\lim_{\epsilon\to 0}\Phi_\epsilon$ exists and

$$\lim_{\epsilon\to 0}\Phi_\epsilon \geq \Phi_0 = \phi_e, \quad dd^c(\lim_{\epsilon\to 0}\Phi_\epsilon) \geq 0$$

Moreover, by definition $\Phi_\epsilon \leq (\phi + \epsilon\phi_A) - \epsilon\phi_A \leq \phi$ on $\mathrm{int}(X)$. Hence $\lim_{\epsilon\to 0}\Phi_\epsilon \leq \phi_e$ by the extremal defintion of ϕ_e. This proves 7.4. Finally, writing

$$(dd^c\Phi_\epsilon)^n/n! = \sum_{k=0}^{n}(-\epsilon)^{(n-k)}(dd^c(\phi + \epsilon\phi_A)_e)^k/k!(dd^c\phi_A)^{(n-k)}/(n-k)!$$

and using that the operator that takes ψ to the current $(dd^c\psi)^k)$ is continuous [26] when applied to a decreasing limit of plurisubharmonic functions (here given by a local representation of $(\phi + \epsilon\phi_A)_e$), proves the last statement of the lemma. $\square$

Now we can prove theorem 1.4 stated in the introduction, saying that the metric on L induced by the Bergman kernel converges to ϕ_e.

Proof of theorem 1.4. Let us first prove the upper bound on $k^{-1}\ln K^k$ if μ is a measure with the Bernstein-Markov property. To this end fix $\epsilon > 0$ and observe that by the very definition of the latter property and the extremal caractererization 2.9 of the Bergman kernel

$$k^{-1}(\ln K^k(x,x) - \ln C_\epsilon) - \epsilon \leq \phi$$

for $x \in X$, for any k. In particular,

$$k^{-1}(\ln K^k(y,y) - \ln C_\epsilon) - \epsilon \leq \phi_e \tag{7.5}$$

on all of Y, by the extremal definition of ϕ_e, which proves the upper bound corresponding to 1.8.

Next, let us prove the lower bound in the case (i). Given $\epsilon > 0$ fix an arbitrary point y in Y. A standard regularization argument involving Demailly's regularization theorem (see [9]) yields a candidate ϕ' for the sup defining ϕ_e such that ϕ' is continuous on X and $\phi'(y) \geq \phi_e(y) - \epsilon$. Now take an element α_k in $H^0(Y, L^k \otimes A)$ furnished by lemma 7.2. Since by construction $\phi' \leq \phi$ on X we have that

$$\int_X |\alpha_k|^2_{k\phi+\phi_A}\,\omega_n \leq \int_Y |\alpha_k|^2_{k\phi'+\phi_A}\,\omega_n \tag{7.6}$$

Hence, 7.3 in lemma 7.2 gives that

$$k^{-1}\ln\left(|\alpha_k(y)|^2_{k\phi'+\phi_A}/\int_X|\alpha_k|^2_{k\phi+\phi_A}\,\omega_n\right)\geq C'k^{-1}$$

and since $\phi'(y)\geq\phi_e(y)-\epsilon$ we then obtain

$$k^{-1}\ln\left(|\alpha_k(y)|^2/\int_X|\alpha_k|^2_{k\phi+\phi_A}\,\omega_n\right)\geq\phi_e(y)-\epsilon+(\phi_A(y)+C')k^{-1}$$

Finally setting $(A,\phi_A)=(L^{k_0},k_0\phi)$ for k_0 a fixed large natural number and writiting $L^k=L^{k-k_0}\otimes A$ proves the lower bound corresponding to 1.8 in the case (i).

The case (ii) now follows from the previous lower bound and the fact that the Bernstein-Markov inequality used to get the upper bound may be replaced by the following stronger inequality if X is a smooth pseudoconcave domain:

$$\sup_{x\in X}|\alpha_k(x)|^2_{k\phi}/\int_X|\alpha_k|^2_{k\phi}\,\omega_n\leq Ck^{(n+1)} \tag{7.7}$$

uniformly in k. Indeed, the previous bound is a direct consequence of the Morse inequalities in section 4 (see also [4] for the "middle region").

To prove the lower bound in the case (iii), fix $\epsilon>0$ and denote by X_δ the closure of an open $\delta-$neighbourhood of X. By the previous argument used to prove the lower bound in the case (ii) applied to X_δ (for δ sufficently small) and with $\phi'=\phi_e$ (which is usc by assumption) it is enough to prove that

$$\int_X|\alpha_k|^2_{k\phi_e}\,\nu\leq C_\epsilon e^{k\epsilon}\int_{X_\delta}|\alpha_k|^2_{k\phi_e}\,\omega_n. \tag{7.8}$$

But this is a simple consequence of the submean property of holomorphic functions. Indeed, for any fixed x in X the latter property gives

$$|\alpha_k|^2(x)\leq C_\delta\int_{B_\delta(x)}|\alpha_k|^2\,\omega_n,$$

in a fixed trivialization of L on the coordinate ball $B_\delta(x)$ of radius δ centered at x. Now since, by assumption, ϕ_e is usc we may chose δ sufficiently small that

$$|\alpha_k|^2_{k\phi_e}(x)\leq C_{\delta(\epsilon)}e^{k\epsilon}\int_{B_\delta(x)}|\alpha_k|^2_{k\phi_e}\,\omega_n(\leq C_{\delta(\epsilon)}e^{k\epsilon}\int_{X_\delta}|\alpha_k|^2_{k\phi_e}\,\omega_n) \tag{7.9}$$

Hence, integrating over x proves 7.8.

Finally, to prove the convergence 1.9 in the theorem recall that the Monge-Ampere operator (mapping ϕ to $(dd^c\phi)^n$) is continuous when applied to a uniform limit [26] of plurisubharmonic functions.

Remark 7.4. Given a compact subset X and a continuous metric ϕ on an ample line bundle L, let

$$\psi_k(y) := \ln\left(\sup_{\alpha_k \in H^0(Y,L^k).} \frac{|\alpha_k(y)|^2}{\sup_Y |\alpha_k(y)|^2_{k\phi}} \right),$$

which is an L^∞-version of the kth Bergman metric. By definition $\psi_k \leq \phi_e$ on X for any k. Hence, the proof of the lower bound in Theorem 1.4 (iii) in gives that ψ_k also converges uniformly to ϕ_e if (X, ϕ) is regular. Moreover, a slight modification of the argument gives that if X is *any* compact set, then the corresponding *point-wise* convergence always holds. Indeed, given a fixed point y in Y and $\epsilon > 0$ one applies lemma 7.2 to a continuous metric ϕ' defined as in the proof of (i) in Theorem 1.4. Then the convergence will in general only be point-wise since the δ in the estimate 7.9 will depend on the "oscillation" of ϕ' (which plays the role of ϕ_e) and hence on the point y.

Proof of corollary 1.5. First assume that L is ample. Then combining theorem 1.3 and theorem 1.4 immediately proves the corollary. When L is not ample we replace it by the ample line bundle $L^m \otimes A$. Finally, letting m tend to infinity and combining the ample case with lemma 7.3 then finishes the proof of the corollary.

Example 7.5. Under the assumption 2 in section 2.2 , we have that $\phi_e \equiv 0$ on $Y - X$ (also assuming that L is a positive line bundle). Indeed, in this case ϕ_e is continuous on all of Y (since it is the uniform limit of continuous functions according to theorem 1.4). Hence, $\phi_e \equiv 0$ on $Y - X$ by the uniqueness of solutions to corresponding Dirichlet type problem.

In particular, the previous example shows that in the case considered in example 3.2 ϕ_e corresponds to the function $\ln(|\zeta|^2_+)$ in $\mathbb{C}^n$. [9] This also follows from the following class of examples:

Example 7.6. In the case considered in example 3.4 we have that

$$\phi_e(z, w) := \phi_G(z) + \rho_+$$

on $Y - X$, using the notation $f_+ = f$ when $f \geq 0$ and $f_+ = 0$ otherwise. In fact, this is the extension of ϕ described in that example and its regularization also described there is incidentally very similar to the one obtained from the limit of $k^{-1}(\ln K^k(y, y)$ (which may be computed following the constant curvature case in section 4 in [4]). Note that ϕ_e is continuous up to the boundary of $Y - X$ and solves the Dirichlet type problem in formula **??**.

Remark 7.7. In the more general setup of Guedj-Zeriahi [25] the pseudoconcave domain X is replaced by any given Borel set K in Y. Moreover, the function $V_{K,\omega} := \phi_e(y) - \phi(y)$, where $\omega = dd^c\phi$, is called the *global extremal function* in [25] and it is an example of an $dd^c\phi-$*plurisubharmonic*

[9]This is the *Siciak-Zaharjuta extremal function* of the unit-ball (also called the *pluricomplex Green function with a pole at infinity* [26]).

function. In particular, the (generalized) equilibrium measure introduced above may be written as $(dd^c\phi_e)_n = (\omega + dd^c V_{K,\omega})_n$.

8. Open problems

In view of Theorem 5.1, concerning a polarized domain X with compatible curvatures, it is natural to make the following conjecture :

Conjecture 8.1. *Suppose that (X, ϕ) is regular and that the measure ν has the Bernstein-Markov property with respect to (X, ϕ). Then the normalized Bergman measure $k^{-n}B^k\nu$ converges weakly to the the equilibrium measure $(dd^c\phi_e)_n$ associated to (X, ϕ).*

If the Bergman measure in the conjecture above is replaced by the normalized volume form $(dd^c(k^{-1}\ln K^k(y, y))_n$ of the corresponding Bergman metric (compare section 6) then the statement does hold, according to theorem 1.4. Hence, the conjecture above is equivalent to the following

Conjecture 8.2. *Under the assumptions of the previous conjecture the weak limits of $k^{-n}B^k(y)\nu$ and $(dd^c(k^{-1}\ln K^k(y, y))_n$ both exist and coincide.*

In the "weighted classical setting" in $\mathbb{C}^n$ the conjecture 8.1 was made independently very recently by Bloom-Levenberg in [17]. In [16] Bloom-Levenberg proved the conjecture for $n = 1$, i.e. in the complex plane, by using the fact that in this case the equilibrium measure may be characterized as a minimizer of the weighted logarithmic energy (see [16] for further references concerning the planar case). When $X = Y$, the metric ϕ is a smooth metric and μ is a smooth volume form the conjecture was proved very recently in [16] (with L *any* line bundle over X).[10] By reducing to this case the conjecture can be shown to hold when X is any disc subbundle of a $\mathbb{P}^1$-bundle (as in section 3, but without any curvature assumptions). The proof will be given in [7].

Returning to the case when X is a polarized pseudoconcave domain and ν a volume form note that if conjecture 8.1 holds than one obtains the following bound on the corresponding equilibrium measure from the local Morse inequalities in section 4:

$$(dd^c\phi_e)_n \leq 1_X(dd^c\phi)_n + [\partial X] \wedge \mu$$

Finally, it seems also natural to conjecture that the factors $C_\epsilon e^{k\epsilon}$in 7.2 may be replaced by Ck^{n+1} for some constant C if X is smooth domain in Y and ϕ is a smooth metric on L. See [39] for results in this direction in the "classical setting". As shown in the following appendix the latter conjecture does hold when X is pseudoconcave. One can also ask if the equilibrium measure of a polarized domain with smooth boundary is such that the measure $(dd^c\phi_e)_n - 1_X(dd^c\phi)_n$ which is supported on ∂X is absolutely continuous w.r.t the "surface measure" on ∂X?

[10] In fact, a much stronger convergence result was obtained in this case, by showing that the corresponding equilibrium measure is absolutely continuous with respect to the Lebesgue measure and is equal to the limit of $k^{-n}B^k$ almost everywhere on X.

9. Appendix

Boundary estimates. In this section we will give the proof of some boundary estimates for the scaled Bergman function $B^{(k)}$, refered to in section 4.3. The arguments are essentially contained in [4], but for completeness we provide some elemantary arguments, which don't use any subelliptic estimates (as opposed to [4]). The notation in section 4.3 will be used, but for notational convenience we assume that the curvature eigenvalues μ_i are all equal to -1. Moreover, note that on X_k the weighted norm is equivalent to the unweighted one: for any function f we have

$$(1/C)\left\|f\right\|^2_{X_k} \leq \left\|f\right\|^2_{X_k,k\phi} \leq C\left\|f\right\|^2_{X_k},$$

which follows immediately from the convergence of the scaled metrics. The following lemma uses the pseudoconcavity of X_0 to estimate the values of a holomorphic function f on a polydisc centered at 0, with the norm of f inside X_k.

Lemma 9.1. *Let f be a holomorphic function on X_k. Then*

$$\sup_{\Delta_R} |f|^2 \leq C_R \left\|f\right\|^2_{X_k}$$

Proof. By a scaling argument we may asume that $R = 1$. First observe that, by the submean property of holomorphic functions,

$$\sup_{\Delta} |f|^2 \leq C \left\|f\right\|^2_{\Delta}.$$

Hence, it is enough to prove that the integral outside X_k may be estimated by an integral inside of X_k :

$$\left\|f\right\|^2_{\Delta - X_k} \leq C' \left\|f\right\|^2_{X_k}. \tag{9.1}$$

The latter estimate is essentially a well-known consequence of the pseudoconcavity of X. To see this, note that for any given point (z_0, w_0) in Δ we have that

$$S_0 := \{(z, w_0) : 3 > |z - z_0| > 2\} \subseteq X_k \tag{9.2}$$

for k large. Indeed, by the uniform convergence 4.10 it is enough to prove the inclusion into X_0, which in turn follows from the bound

$$\rho_0(z, w_0) := \mathrm{Im} w_0 - |z|^2 \leq 1 - |z|^2 < 0,$$

if (z, w_0) is in S_0. Next note that by the submean-property of the holomorphic function $f(\cdot, w_0)$ we have

$$|f|^2 (z_0, w_0) \leq C \frac{\int_2^3 (\int_{|z|=r} |f|^2 (z, w_0) d\widehat{\sigma}) r^{2n-1} dr}{\int_2^3 r^{2n-1} dr} = C' \int_{S_0} |f|^2 (z, w_0) dz \wedge d\bar{z},$$

where $d\widehat{\sigma}$ is the normalized measure on the sphere in $\mathbb{C}^{n-1}_z$ of radius r centered at z_0. Finally the bound 9.1 is obained by first integrating over the $z-$variable in the left hand side of 9.1 and then using the previous point-wise estimate on the integrand. The point is that, by 9.2, S_0 is a subset of X_0. $\square$

The next lemma is independent of any curvature assumptions.

Lemma 9.2. *Let f be a holomorphic function on X_k. Then for $v \in [-\frac{1}{2}\ln k, 0]$ we have that*

$$\left|f(0,iv)\right|^2 \leq \frac{C}{v^2}\left\|f\right\|_{X_k}^2$$

Proof. By the submean propery of the holomorphic function $f(0,\cdot)$ we have

$$\left|f(0,iv)\right|^2 \leq \frac{C}{v^2}\int_{|w-iv|\leq -v/2}\left|f\right|^2(0,w)dw\wedge d\bar{w}.$$

Note that, by assumption, the integration takes place over points inside X_k. Finally, estimating $\left|f\right|^2(0,w)$, using the submean property of $f(\cdot,w)$ over the unit-ball in the $z-$variables, then finishes the proof of the lemma. □

Now by combining the two previous lemmas we obtain that the function $1_{R_k\leq v\leq 0}(v)B^{(k)}(0,iv)$ is dominated by the following function which is in $L^1]-\infty,0[$:

$$g(v) := C, \text{for } v \geq -1 \quad g(v) := \frac{C}{v^2}\text{ for } v < -1$$

References

[1] Dai, X; Liu, K- Ma, X: On the asymptotic expansion of the Bergman kernel, J. Differential Geom. 72 (2006), no. 1, 1–41

[2] Berman, R: Bergman kernels and local holomorphic Morse inequalities. Math Z., Vol 248, Nr 2 (2004), 325–344

[3] Berman, R: Super Toeplitz operators on holomorphic line bundles J. Geom. Anal. 16 (2006), no. 1, 1–22.

[4] Berman, R., Holomorphic Morse inequalities on manifolds with boundary Ann. Inst. Fourier (Grenoble) 55 (2005), no. 4, 1055–1103.

[5] Berman, R., Correction to "Holomorphic Morse Inequalities on Manifolds with Boundary". To appear in Ann. Inst. Fourier (Grenoble)

[6] Berman, R., Bergman kernels and equilibrium measures for line bundles over projective manifolds. Preprint in 2007 at arXiv:0710.4375

[7] Berman, R, Random measure processes on complex manifolds in the presence of negative curvature (in preparation)

[8] Berman R; Berndtsson B; Sjöstrand J: Asymptotics of Bergman kernels. Preprint at arXiv.org/abs/math.CV/050636.

[9] Berman R; Boucksom S: Capacities and weighted volumes for line bundles. In preparation.

[10] Berndtsson, Bo: Positivity of direct image bundles and convexity on the space of Kähler metrics. Preprint at arXiv.org/abs/math.CV/0608385

[11] Bleher, Pl; Shiffman, B; Zelditch, S: Universality and scaling of correlations between zeros on complex manifolds. Invent. Math. 142 (2000), no. 2, 351–395.

[12] Bleher, Pl; Shiffman, B; Zelditch, S: Poincaré-Lelong approach to universality and scaling of correlations between zeros. Comm. Math. Phys. 208 (2000), no. 3, 771–785.

[13] Bloom,T: Random polynomials and Green functions. Int. Math. Res. Not. 2005, no. 28, 1689–1708.

[14] Bloom,T: Random polynomials and (pluri)potential theory. Anneles Polonici Math. 91.2-3 (2007)
[15] Bloom, T; Levenberg, N: Weighted pluripotential theory in $\mathbb{C}^N$ Amer. J. Math. 125 (3) (2003), 57-103
[16] Bloom, T; Levenberg, N: Strong asymptotics for Christoffel functions of planar measures. Preprint in 2007 at ArXiv: 0709.2073
[17] Bloom, T; Levenberg, N: Transfinite diameter notions in C^N and integrals of Vandermonde determinants. Preprint in 2007 at ArXiv: 0712.2844
[18] Bloom, T; Shiffman, B: Zeros of random polynomials on $\mathbb{C}^m$ Preprint at arXiv.org/abs/math.CV/0605739
[19] Deift, P. A. Orthogonal polynomials and random matrices: a Riemann-Hilbert approach. Courant Lecture Notes in Mathematics, 3. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.
[20] Demailly, J-P: Champs magnetiques et inegalite de Morse pour la d”-cohomologie., Ann Inst Fourier, 355 (1985,185-229)
[21] Demailly, J-P: Complex analytic and algebraic geometry. Available at www-fourier.ujf-grenoble.fr/~demailly/books.html
[22] Demailly, J-P: Potential Theory in Several Complex Variables. Manuscript available at www-fourier.ujf-grenoble.fr/~demailly/
[23] Griffiths, P; Harris, J: Principles of algebraic geometry. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.
[24] Gromov, M: Convex sets and Kähler manifolds, Advances in Differential Geometry and Topology (Teaneck, NJ), World Scientific Publishing, 1990, pp. 1-38.
[25] Guedj,V; Zeriahi, A: Intrinsic capacities on compact Kähler manifolds. J. Geom. Anal. 15 (2005), no. 4, 607–639.
[26] Klimek, M: Pluripotential theory. London Mathematical Society Monographs. New Series, 6. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991
[27] Lazarsfeld, Robert: Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. A series of modern surveys in Mathematics, 48. Springer-Verlag, Berlin, 2004
[28] Lindholm, N: Sampling in weighted L^p spaces of entire functions in $\mathbb{C}^n$ and estimates of the Bergman kernel. J. Funct. Anal. 182 (2001), no. 2, 390–426
[29] Phong, D.H;Sturm, J: The Monge-Ampere operator and geodesics in the space of Kähler potentials. Invent. Math. 166 (2006), no. 1, 125–149.
[30] Saff.E; Totik.V: Logarithmic potentials with exteriour fields. Springer-Verlag, Berlin. (1997)
[31] Shiffman, B; Zelditch, S: Distribution of zeros of random and quantum chaotic sections of positive line bundles. Comm. Math. Phys. 200 (1999), no. 3, 661–683.
[32] Shiffman, B; Zelditch S: Equilibrium distribution of zeros of random polynomials. Int. Math. Res. Not. 2003, no. 1, 25–49.
[33] Shiffman, B; Zelditch, S: Random polynomials with prescribed Newton polytope. J. Amer. Math. Soc. 17 (2004), no. 1, 49–108
[34] Shiffman, B; Zelditch, S: Number variance of random zeros on complex manifolds. Preprint at arXiv.org/abs/math.CV/0608743
[35] Tian, G: On a set of polarized Kähler metrics on algebraic manifolds. J. Differential Geom. 32 (1990), no. 1, 99–130
[36] Witten, E: Supersymmetry and Morse theory. J. Differential Geom. 17 (1982), no. 4, 661–692.
[37] Zabrodin, A; Matrix models and growth processes: from viscous flows to the quantum Hall effect. Preprint in 2004 at arXiv.org/abs/hep-th/0411437

[38] Zelditch, S: Szegö kernels and a theorem of Tian. Internat. Math. Res. Notices 1998, no. 6, 317–331.

[39] Zeriahi,A: Inegalites de Markov et developpement de serie de polynomes orthogonaux des fonctions $\mathcal{C}^{\infty}$ et $\mathcal{A}^{\infty *}$. In: Several Complex Variables Proceedings of the Mittag-Leffler Inst. 1987-88. Mathematical Notes (1993),683–701. Princ. Univ. Press.

Current address: Institut Fourier, 100 rue des Maths, BP 74, 38402 St Martin d'Heres (France)

E-mail address: robertb@math.chalmers.se